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ELEMENTARY ALGEBRA

BY

CHARLES SMITH M.A.

FELLOW AND TUTOR OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

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1886

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PREFACE.

IN the following work I have endeavoured to explain the principles of Algebra in as simple a manner as possible for the benefit of beginners, bestowing great care upon the explanations and proofs of the fundamental operations and rules. The book will be found to differ considerably from those at present in use, but I do not think the new methods will prove to be difficult.

Writers of elementary text books, in attempting to smooth the path of the learner, are exposed to the danger of paying too little attention to the logical ground work of their subject: an apparent simplicity is thus gained, but at a loss of much of the educational value. I have endeavoured to avoid this very real and serious danger.

The book is intended to be a fairly complete treatise on Elementary Algebra up to the Binomial Theorem for a positive integral exponent: series and expansions which introduce difficulties as to convergency are not included in its range.

I hope to publish shortly a more complete treatise on Algebra, to which the present book would serve as an introduction.

The examples have been carefully selected and arranged so as to illustrate and enforce all important principles; and they are sufficiently numerous to give ample exercise in the manipulation of algebraical expressions. To ensure sufficient variety I have consulted nearly all the examination papers which have been set in Cambridge for many years past; and I have also, by the kind permission of the Local Examinations Syndicate, been enabled to make use of the questions which have been given in the Local Examinations of the University. The sets of Miscellaneous Examples which are placed at intervals throughout the book will, I believe, be found of great value.

I am indebted to several of my friends for their kindness in looking over the proof sheets, and in working the examples; and also for many valuable suggestions. My especial thanks are due to Mrs Bryant, D.Sc., and to Mr R. H. Piggott, B.A.

CHARLES SMITH.

SIDNEY SUSSEX COLLEGE,
January, 1886.

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ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. **ALGEBRA**, like Arithmetic, is a science which treats of *numbers*.

In Arithmetic numbers are represented by *figures*, each of which has one and only one meaning.

In Algebra numbers are represented either by figures, as in Arithmetic, or by the *letters of the alphabet*.

Usually each letter may be supposed to represent any number whatever; but, in any connected series of operations, each letter must throughout be supposed to represent the same number.

2. The numbers treated of may be either whole numbers or fractions. Whole numbers are often called *integers*. All concrete quantities, such as values, lengths, areas, periods of time, &c., with which we have to do, must be measured by the *number of times* each contains some unit of its own kind. Thus, we have lengths of 4, $\frac{2}{3}$, $5\frac{1}{4}$, the unit being an inch, a yard, a mile, or any other fixed length. It is only these *numbers* with which we are concerned, and our symbols of quantity, whether figures or letters, always represent numbers.

3. In Algebra we make use of all the signs which are used in Arithmetic; but, as we shall see later, it is necessary to extend the meaning of some of the signs. We proceed to give definitions of the principal signs.

4. The sign +, which is read "plus," is placed before a number to indicate that it is to be *added* to what has gone before. Thus $6+3$ means that 3 is to be added to 6; $6+3+2$ means that 3 is to be added to 6, and then 2 added to the result; so also, $a+b$ means that the number which is represented by b is to be added to the number which is represented by a; or, expressed more briefly, it means that b is to be added to a.

5. The sign −, which is read "minus," is placed before a number to indicate that it is to be *subtracted* from what has gone before. Thus $6-3$ means that 3 is to be subtracted from 6; $6-3-2$ means that 3 is to be subtracted from 6, and then 2 subtracted from the result; $6-3+2$ means that 3 is to be subtracted from 6, and then 2 added to the result; so also, $a-b$ means that b is to be subtracted from a.

6. The sign ×, which is read "into," is placed between two numbers to indicate that the first number is to be *multiplied* by the second. Thus 6×3 means that 6 is to be multiplied by 3; so also $a\times b$ means that a is to be multiplied by b.

The sign × is however generally omitted when the numbers are expressed by means of letters, and the letters are placed consecutively. Thus ab means the same as $a\times b$, and abc the same as $a\times b\times c$.

The × is also unnecessary between a figure and a letter. Thus $3\times a$ is written $3a$.

We cannot however omit the sign of multiplication between figures: 63, for example, has already a meaning assigned to it, namely *sixty-three*, which is different from 6×3.

Sometimes the × is replaced by a point, which is placed *on* the line, to distinguish it from the decimal point, which is placed *above* the line. Thus $a \times b \times c$, $a \,.\, b \,.\, c$ and abc all mean that a is to be multiplied by b and the result multiplied by c.

7. The sign ÷, which is read "divided by," or "by," is placed between two numbers to indicate that the first number, called the dividend, is to be *divided* by the second number, called the divisor. Thus $6 \div 3$ means that 6 is to be divided by 3; so also $a \div b$ means that a is to be divided by b.

The operation of division is often indicated by placing the dividend over the divisor with a line between them: thus $\frac{a}{b}$ is used instead of $a \div b$.

Note. It is important for the student to notice the *order* of the operations in such an expression as $a + b \times c$. This means that b is first to be multiplied by c, and the result added to a. So also $a - b \div c$ means that b is first to be divided by c, and the result subtracted from a.

8. When two or more numbers are multiplied together the result is called the *continued product*, or the *product;* and each number is called a *factor* of the product.

When the factors of the product are considered as divided into two sets, each is called the *coefficient*, that is the *co-factor*, of the other. Thus, in $3abx$, 3 is the coefficient of abx, and abx is the coefficient of 3; also $3a$ is the coefficient of bx, and $3ab$ is the coefficient of x.

When one of the factors of a product is a number expressed in figures, it is called the *numerical coefficient* of the product of the other factors.

9. When a product consists of the same factor repeated any number of times it is called a *power* of that factor. Thus aa is called the second power of a, aaa is called the third power of a, $aaaa$ is called the fourth power of a, and so on. Sometimes a is called the first power of a.

Special names are also given to aa and aaa; they are called respectively the *square* and the *cube* of a.

10. Instead of writing aa, aaa, etc., a more convenient notation is adopted as follows: a^2 is used instead of aa, a^3 is used instead of aaa, and a^n is used instead of $aaa\ldots\ldots$, the factor a being taken n times; the small figure placed above and to the right of a, shewing the number of times the factor a is to be taken. So also a^3b^2 is written instead of $aaabb$, and similarly in other cases.

The small figure, or letter, placed above a symbol to indicate the number of times that symbol is to be taken as a factor is called the *index* or the *exponent*. Thus a^n means that the factor a is to be taken n times, or that the nth *power* of a is to be taken, and n is called the *index*.

When the factor a is only to be taken once, we do not write it a^1, but simply a.

11. The quantity which when squared is equal to any number a is called the *square root* of a, and is represented by the symbol $\sqrt[2]{a}$, or more often by $\sqrt{a}$: thus 2 is $\sqrt{4}$, for 2^2 is 4.

The quantity which when cubed is equal to any number a is called the *cube root* of a, and is represented by the symbol $\sqrt[3]{a}$: thus $3 = \sqrt[3]{27}$, for $3^3 = 27$.

In general, the quantity which when raised to the nth power, where n is any whole number, is equal to any number a, is called the nth root of a, and is represented by the symbol $\sqrt[n]{a}$.

The sign $\sqrt{\ }$ was originally the initial letter of the word *radix*. It is often called the *radical* sign.

A root which cannot be obtained exactly is called a *surd*, or an *irrational* quantity: thus $\sqrt{7}$ and $\sqrt[3]{4}$ are surds.

The approximate value of a surd, for example of $\sqrt{7}$, can be found, to any degree of accuracy which may be desired, by the ordinary arithmetical process; but we are not required to find these approximate values in Algebra: for us $\sqrt{7}$ is simply that quantity which when squared will become 7.

12. The sign $=$, which is read "equals," or "is equal to," is placed between two quantities to indicate that they are *equal* to one another. Thus $5+7=12$, which is read five plus seven equals twelve.

The sign $>$ indicates that the number which precedes the sign *is greater than* that which follows it. Thus $a>b$ means that a is greater than b.

The sign $<$ indicates that the number which precedes the sign is *less than* that which follows it.

Thus $a<b$ means that a is less than b.

The sign $\because$ is written for the word *because* or *since*.

The sign $\therefore$ is written for the word *therefore* or *hence*.

13. A collection of algebraical symbols, that is of letters, figures, and signs, is called an *algebraical expression*.

The parts of an algebraical expression which are connected by the signs + or − are called the *terms*.

Thus $2a-3bx+5cy^2$ is an algebraical expression containing the three terms $2a$, $-3bx$ and $+5cy^2$.

14. When two terms contain the same letters every one of which is raised to the same power in both, they are called *like* terms. Thus $3ab^2x^3$ and $7ab^2x^3$ are *like* terms; but, although $3a^2bx^3$ and $7a^2b^2x^3$ contain the same letters, they are not like terms, for all the letters are not raised to the same power.

15. A *simple* expression is one which contains only one term, and a *compound* expression is one which contains more than one term. Thus $5ab^2cx$ is a simple expression, and $a+b$ is a compound expression.

An expression which consists of *two* terms is called a *binomial* expression; an expression which consists of *three* terms is called a *trinomial* expression; and an expression which consists of more than three terms is called a *multinomial* expression.

16. To denote that an algebraical expression is to be treated as a whole, it is put between brackets. Thus $(a+b)\,c$ means that we are to add a and b, and then multiply the result by c; if we were to omit the brackets and write $a+bc$ it would mean that b was to be multiplied by c and then the result added to a.

Brackets are of various shapes: thus ( ), { }, [ ]. Instead of a pair of brackets a line, called a *vinculum*, is often drawn over the numbers which are to be treated as a whole: thus $a-\overline{b-c}$ is equivalent to $a-(b-c)$, and $\sqrt{a+b}$ is equivalent to $\sqrt{}(a+b)$. When no vinculum or bracket is used, a radical sign refers only to that quantity which immediately follows it: thus $\sqrt{}2a$ means that the square root of 2 is to be multiplied by a, whereas $\sqrt{2a}$ means that we are to take the square root of $2a$; also $\sqrt{a+x}$ means the square root of the sum of a and x, but $\sqrt{}a+x$ means the sum of x

and the square root of a. The line between the numerator and denominator of a fraction acts as a vinculum, for $\dfrac{a+b}{3}$ is the same as $(a+b) \div 3$.

EXAMPLES. I.

1. Calculate the values of the following :—

(i) $7-5+4$, (ii) $2+7\times 8+1$, (iii) $2+3\times 5-4\div 3+\frac{1}{3}$.

2. If $a=1$, $b=2$, $c=3$, find the values of

(i) $15a-bc$, (ii) $ab+c$, (iii) $7ab-b-c$, (iv) $3ab-2c$,
(v) $5a+2b-3c$ and (vi) $bc+ca+ab$.

3. Write down the values of 2^4, 3^3, 4^3, $\sqrt{64}$, $\sqrt[3]{64}$ and $\sqrt[4]{16}$.

4. If $a=3$, $b=\frac{1}{2}$, $c=1$, find the values of

(i) $4b^2+a^2-3c^3$, (ii) $5ab-10bc^2$,
(iii) c^3-8b^3 and (iv) $ab^2c^3-\frac{1}{3}a^3b^2c$.

5. Shew that x^2-5x+6 is equal to zero, if $x=2$, or if $x=3$.

6. Shew that $2x^3-11x^2+17x-6$ is equal to zero, if $x=2$, or if $x=3$, or if $x=\frac{1}{2}$.

7. Calculate $(2+3)\,4-3\,(5-1)$, and $(5-2)^2+3-2^2$.

8. Find the values of the following expressions in which $a=1$, $b=2$, $c=3$:—

(i) $23a-2\,(b+2c)$, (ii) $(2a-b)^2+4\,(2c-b)^3$,
(iii) $3\,(a+b)^2-(b+c)^2$, and (iv) $6\,(a+b)\,c-(b+c)\,abc$.

9. Find the values of

$$a^3-b^3,\ (a-b)\,(a^2+ab+b^2),\ (a-b)^3+3ab\,(a-b)$$

and

$$(a+b)^3-3ab\,(a+b)-2b^3,$$

when $a=5$, $b=3$; and also when $a=4$, $b=2$.

10. Find the values of

$$\sqrt{a^2-b^2},\quad \sqrt{5a},\quad \sqrt{2bc}+a,\quad \sqrt{bc+c^2},\quad \text{and}\quad \sqrt[3]{(2a^2+b^2-8c^2)},$$

when $a=5$, $b=4$, and $c=\frac{1}{2}$.

CHAPTER II.

NEGATIVE QUANTITIES. FUNDAMENTAL LAWS.

17. BEFORE considering the fundamental laws of algebra, we must explain what is meant by *positive* and *negative* quantities.

18. Whatever kind of quantity we are considering + 4 will stand for what *increases* that quantity by 4 units, and − 4 will stand for whatever *decreases* the quantity by 4 units.

If we are calculating the amount of a man's property (estimated in pounds), + 4 will stand for whatever increases his property by £4, that is, + 4 stands for £4 that he possesses, or that is owing to him; so also − 4 will stand for whatever decreases his property by £4, that is, − 4 will stand for £4 that he owes.

If, on the other hand, we are calculating the amount of a man's debts, + 4 will stand for whatever increases his debts, that is, + 4 will now stand for a debt of £4; so also − 4 will now stand for whatever decreases his debts, that is, − 4 will stand for £4 that he has, or that is owing to him.

If we are calculating the amount of a man's gains, + 4 will stand for what increases his total gain, that is, + 4 will stand for a *gain* of 4; so also − 4 will stand for what decreases his total gain, that is, − 4 will stand for a *loss* of 4.

If however we are calculating the amount of a man's losses, + 4 will stand for a *loss* of 4, and − 4 will stand for a *gain* of 4.

Again, if the magnitude to be increased or diminished is the height of a barometer, + 4 will stand for a *rise* of 4 units, and − 4 will stand for a *fall* of 4 units.

If the magnitude to be increased is the distance from any place, measured in any particular direction, + 4 will stand for a distance of 4 units in that direction, and − 4 will stand for a distance of 4 units in the *opposite* direction.

19. We might consider many other examples; but we have given sufficient to shew that the signs + and −, in addition to their uses as signs of the *operations* of addition and subtraction, will serve to distinguish between magnitudes of opposite kinds. Thus, whatever + 4 may represent, − 4 will represent an equal magnitude, but of the opposite kind; if, for example, + 4 represent a gain of £4, − 4 will represent a loss of £4; if + 4 represent a fall of 4 inches, − 4 will represent a rise of 4 inches; if + 4 represent a distance of 4 miles measured to the East, − 4 will represent a distance of 4 miles measured to the West; and so on, for all other kinds of contrariety.

20. A quantity to which the sign + is prefixed is called a *positive* quantity, and a quantity to which the sign − is prefixed is called a *negative* quantity.

The signs + and − are called the *positive* and *negative* signs. They are also sometimes called *signs of affection*, when they are used to indicate a *quality* of the quantities before whose symbols they are placed.

The sign + as a sign of affection is frequently omitted: thus 4 is written instead of + 4. When neither the + nor the − sign is prefixed to a number, the + sign is to be understood.

NOTE. Although there are many signs used in algebra, the name *sign* is often used to denote the two signs + and − exclusively. When the sign of a quantity is spoken of, it means the + or − sign which is prefixed to it; and when we are directed to change the signs of an expression, it means that we are to change + and − wherever they occur into − and + respectively.

21. The magnitude of a quantity considered independently of its quality, or of its sign, is called its *absolute* magnitude. Thus a rise of 4 feet and a fall of 4 feet are equal in absolute magnitude; so also $+4$ and -4 are equal in absolute magnitude, whatever the unit may be.

22. The process of finding the result when two or more quantities are taken together is called *addition*, and the result is called the *sum*.

Since a positive quantity produces an increase, and a negative quantity produces a decrease, to add a positive quantity we must *add* its absolute value, and to add a negative quantity we must *subtract* its absolute value.

Thus, when we add $+4$ to $+6$, we get $+6+4=+10$; and when we add -4 to $+10$, we get $+10-4=+6$.

So also, when we add $+b$ to a, we get $a+b$; and when we add $-b$ to a, we get $a-b$. Hence

$$a+(+b)=a+b,$$
$$a+(-b)=a-b$$

We therefore have the following rule for the addition of any term:—*To add any term affix it to the expression to which it is to be added, with its sign unchanged.*

When numerical values are given to a and to b, the numerical values of $a+b$ and $a-b$ can be found; but until we know what numbers a and b represent, we cannot take any further step, and the process is considered to be algebraically complete.

23. It should be noticed that when b is greater than a, the arithmetical operation denoted by $a-b$ is impossible; for we cannot take any number from a smaller number. Thus, if $a=3$ and $b=5$, $a-b$ will be $3-5$, and we cannot take 5 from 3. But to subtract 5 is the same as to subtract 3 and 2 in succession, so that $3-5=3-3-2=-2$: we then consider that -2 is 2 which is to be subtracted from the other parts of the algebraical expression, or that -2 is two units of the kind opposite to that represented by 2.

24. An example will shew the distinction between addition in Algebra and in Arithmetic.

A boy played two games; in the first game he won 6 points, and in the second he won -4 points (that is, he lost 4). How many did he win altogether?

The total gain in the two games together is what is meant in Algebra by the *sum* of the gains. Hence, to obtain the total gain, we must *add* 6 and -4, and this operation is indicated by $6+(-4)$. Now the total gain is clearly $6-4$. Hence $6+(-4)=6-4$.

Addition in Arithmetic always produces an increase; but this is by no means necessarily the case in Algebra, for the addition of a negative quantity produces a decrease.

25. Subtraction is the reverse of addition. Hence to subtract a positive quantity produces a *decrease*, and to subtract a negative quantity produces an *increase*.

To subtract $+4$ from $+10$, we must decrease the amount by 4; we then get $+10-4=+6$. And to subtract -4 from $+6$, we must increase the amount by 4; we then get $+6+4=+10$.

Hence $\qquad +10-(+4)=+10-4=\ 6,$

and $\qquad +\ 6-(-4)=+\ 6+4=10.$

So also $\qquad +a-(+b)=+a-b,$

and $\qquad +a-(-b)=+a+b.$

We therefore have the following rule for the subtraction of any term :—*To subtract any term affix it to the expression from which it is to be subtracted but with its sign changed.*

26. The rule for subtraction can also be shewn as follows.

Since subtraction is the reverse of addition, an addition and a subtraction of the same quantity produces no effect. Hence a is the same as $a+b-b$. Now if we take $+b$ from $a+b-b$, what is left is $a-b$. So that if we take away $+b$ from a the remainder is $a-b$, that is

$$a-(+b)=a-b.$$

Again, if we take away $-b$ from $a+b-b$, what is left is $a+b$. So that if we take away $-b$ from a the remainder is $a+b$, that is

$$a-(-b)=a+b.$$

27. We have hitherto supposed that the letters used to represent quantities were restricted to positive values. If this restriction were retained Algebra would lose much of its value: in what follows therefore it must always be understood, unless the contrary is expressly stated, that each letter may have any positive or negative value.

Since any letter may stand for either a positive or for a negative quantity, a term preceded by the sign $+$, for example $+a$, is not necessarily a positive quantity; such terms are however generally called *positive terms*, because they are so in outward form; and the terms preceded by the sign $-$ are similarly called *negative terms*.

28. We have already proved [Art. 22 and Art. 25] that when b was any *positive* quantity

$$\left.\begin{array}{ll} a+(+b)=a+b \;\ldots\ldots\ldots\ldots & \text{(i)} \\ a+(-b)=a-b \;\ldots\ldots\ldots\ldots & \text{(ii)} \\ a-(+b)=a-b \;\ldots\ldots\ldots\ldots & \text{(iii)} \\ \text{and} \quad a-(-b)=a+b \;\ldots\ldots\ldots\ldots & \text{(iv)} \end{array}\right\} \ldots\ldots\ldots(a).$$

The above laws being true for all positive values must be true for negative values also. For let $c=-b$, where b is any positive quantity; then c is any negative quantity, and we have

$$+c=+(-b)=-b,$$

and

$$-c=-(-b)=+b.$$

Hence, putting $+c$ for $-b$, and $-c$ for $+b$ in (i), (ii), (iii) (iv), we get

$$\begin{aligned} a+(-c)&=a-c, \\ a+(+c)&=a+c, \\ a-(-c)&=a+c, \\ a-(+c)&=a-c. \end{aligned}$$

But these are the same laws as before, although they are in different order.

Hence the laws expressed in (a) are true for *all values* of b.

29. One quantity a is said to be greater than another quantity b when the difference $a-b$ is positive. In this definition we suppose that the letters may have any values whatever, whether positive or negative.

From the definition it is easy to see that in the series 1, 2, 3, 4, &c., each number is *greater* than the one before it; and that, in the series -1, -2, -3, -4, &c., each number is *less* than the one before it.

Thus 7, 5, 0, -5, -7 are in descending order of magnitude.

EXAMPLES. II.

1. Find the sum of

(i) 4 and -3, (ii) 3 and -4, (iii) -5 and -2, (iv) -3 and -7.

2. Find the sum (i) of 5, -6 and 2; (ii) of -3, 4, -5, and 6; and (iii) of 4, -3, 5 and -6.

3. Subtract (i) 3 from -4, (ii) -4 from 3, (iii) -6 from 8.

4. Subtract $-a$ from $-b$, and find the value of the result when $a=-2$, $b=-3$.

5. A boy won 15 marbles and afterwards lost 27; how many did he win altogether?

6. A man lost £100 in the first year after he began his business, and £50 in the second year; in the third year he gained £120, and in the fourth year he gained £150. How many pounds had he gained at the end of the fourth year, and how many at the end of the third?

7. A barometer fell ·01 inches one day, it rose ·015 inches on the next day, and fell again ·01 inches on the third day. How many inches did it rise altogether in the three days?

8. A train, which was 10 minutes late, arrived at a station at 5 minutes past 12. At what time after noon was it due?

9. A thermometer, which stood at 10 degrees Centigrade, fell 20 degrees when it was put into a freezing mixture: what was the final reading?

10. Which is the greater, the sum of 4 and -2, or the difference of 2 and -1?

11. Which is the greater, the sum of 5 and -1, or the difference of 3 and -2?

12. Find the value of $a-b+c$ and of $-a+b+c$, when $a=1$, $b=-2$ and $c=3$.

13. Find the value of $a+b+c$, and of $a-b+c$, when $a=-2$ $b=-3$ and $c=5$; also when $a=4$, $b=3$, and $c=-1$.

14. Find the value of $a+(-b)+c$ and of $a-(-b)+c$ when $a=3$, $b=-2$ and $c=-1$.

15. Find the value of $-a-(-b)+c$ and of $-(-a)+(-b)-c$, when $a=-1$, $b=-2$ and $c=1$.

30. In Arithmetic, multiplication is first defined to be the taking one number as many times as there are units in another. Thus, to multiply 5 by 4, we take as many fives as there are units in four. As soon, however, as fractional numbers are considered, it is found necessary to modify somewhat the meaning of multiplication, for by the original definition we can only multiply by *whole* numbers. The following is therefore taken as the definition of multiplication: "*To multiply one number by a second we do to the first what is done to unity to obtain the second*".

Thus $\quad 4$ is $1+1+1+1$;

$\therefore \quad 5\times 4$ is $5+5+5+5$.

Again, to multiply $\frac{5}{7}$ by $\frac{3}{4}$, we must do to $\frac{5}{7}$ what is done to unity to obtain $\frac{3}{4}$, that is we must divide $\frac{5}{7}$ into four equal parts and take three of those parts. Each of the parts into which $\frac{5}{7}$ is to be divided will be $\frac{5}{7 \times 4}$, and by taking three of these we get $\frac{5 \times 3}{7 \times 4}$.

So also,
$$\begin{aligned}(-5) \times 4 &= -5 + (-5) + (-5) + (-5)\\ &= -5 - 5 - 5 - 5\\ &= -20.\end{aligned}$$

With the above definition multiplication by a negative quantity presents no difficulty.

For example, to multiply 4 by -5. Since to subtract 5 by one subtraction is the same as to subtract five units successively,

$$-5 = -1 - 1 - 1 - 1 - 1;$$
$$\begin{aligned}\therefore\quad 4 \times (-5) &= -4 - 4 - 4 - 4 - 4\\ &= -20.\end{aligned}$$

Again, to multiply -5 by -4. Since

$$-4 = -1 - 1 - 1 - 1;$$
$$\begin{aligned}\therefore\quad (-5) \times (-4) &= -(-5) - (-5) - (-5) - (-5)\\ &= +5 + 5 + 5 + 5 \qquad \text{[Art. 28.]}\\ &= +20.\end{aligned}$$

We can proceed in a similar manner for any other numbers whether integral or fractional, positive or negative. Hence we have the following laws:

$$\left.\begin{aligned}a \times b &= +ab \ldots\ldots\text{(i)}\\ (-a) \times b &= -ab \ldots\ldots\text{(ii)}\\ a \times (-b) &= -ab \ldots\ldots\text{(iii)}\\ (-a) \times (-b) &= +ab \ldots\ldots\text{(iv)}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(\beta).$$

The rule by which we determine the signs of the products is called the **Law of Signs**: this law is sometimes enunciated briefly as follows:—*like signs give* +, *and unlike signs* −.

31. It is proved in Arithmetic* that when one number, whether integral or fractional, is multiplied by a second, the result is the same as when the second is multiplied by the first. Hence we have $ab = ba$, for all *positive* values of a and b; and the proposition being true for any positive values of a and b, it must be true for all values, whether positive or negative; for the absolute value of the product is independent of the signs, and the sign of the product is by the Law of Signs independent of the order of the factors. Hence for all values of a and b we have

$$ab = ba \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\gamma).$$

It is also proved in Arithmetic that to multiply by any two numbers in succession, gives the same result as to multiply at once by their product; and this being true for

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* The proof is as follows:—when the numbers are integers, a and b suppose, write down a series of rows of units, putting a units in each row; and take b rows, writing the units under one another as below:

$$\begin{array}{llllll} 1 & 1 & 1 & 1 & 1 & 1\ldots\ldots a \text{ in a row} \\ 1 & 1 & 1 & 1 & 1 & 1\ldots\ldots \\ 1 & 1 & 1 & 1 & 1 & 1\ldots\ldots \\ 1 & 1 & 1 & 1 & 1 & 1\ldots\ldots \end{array}$$

b rows.

Then the whole number of the units is a repeated b times, that is $a \times b$. Now consider the columns instead of the rows: there are clearly b units in each column, and there are a columns; thus the whole number of units is b repeated a times, that is $b \times a$. Hence when a and b are integers $ab = ba$.

When the numbers are fractions, for example $\frac{5}{7}$ and $\frac{3}{4}$, we prove as in Article 30 that $\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4}$. And by the above proof for integers $\frac{5 \times 3}{7 \times 4} = \frac{3 \times 5}{4 \times 7}$; hence $\frac{5}{7} \times \frac{3}{4} = \frac{3}{4} \times \frac{5}{7}$.

all positive numbers, it follows from Art. 30 that it is true in all cases. Thus for all values of a, b and c we have

$$a \times b \times c = a \times (bc).$$

By continued application of the above it is easy to shew that the factors of a product may be taken in any order, however many factors there may be. Thus $abc = bac = bca$, &c. Hence $3a \times 4b = 3 \times 4ab = 12ab$; so also

$$3a \times -5a^2 = -3 \times 5aa^2 = -15a^3.$$

Again $$(ab)^2 = abab = aabb = a^2b^2$$

and $$(\sqrt{2a})^2 = \sqrt{2a}\sqrt{2a} = \sqrt{2}\sqrt{2aa} = 2a^2.$$

Although the order of the factors in a product is indifferent, a factor expressed in figures is always put first, and the letters are usually arranged in alphabetical order.

32. Since $a^2 = aa$, and $a^3 = aaa$; we have

$$a^2 \times a^3 = aa \times aaa = aaaaa = a^5 = a^{2+3}.$$

So also $$a^3 \times a^4 = aaa \times aaaa = aaaaaaa = a^7 = a^{3+4};$$

and $$a^4 \times a = aaaa \times a = aaaaa = a^5 = a^{4+1}.$$

In the above examples we see that *the index of the product of two powers of the same letter is equal to the sum of the indices of the factors.* We can prove in the following manner that this is true whenever the indices are positive integers:

Since by definition

$$a^m = aaaa \ldots\ldots \text{ to } m \text{ factors,}$$

and $$a^n = aaaa \ldots\ldots \text{ to } n \text{ factors;}$$

$$\begin{aligned} \therefore\ a^m \times a^n &= (aaa \ldots \text{ to } m \text{ factors}) \times (aaa \ldots \text{ to } n \text{ factors}) \\ &= aaa \ldots\ldots \text{ to } m+n \text{ factors,} \\ &= a^{m+n}, \text{ by definition,} \end{aligned}$$

so that $a^m \times a^n = a^{m+n}$ .... (δ).

The law expressed in (δ) is called the **Index Law.**

33. Since $(-a) \times (-a) = a^2 = (+a) \times (+a)$ [Art. 30], it follows conversely that the square root of a^2 is either $+a$ or $-a$: this is written $\sqrt{a^2} = \pm a$, the double sign being read 'plus or minus.'

Thus the square root of any algebraical quantity has *two* values, equal in absolute magnitude but opposite in sign.

EXAMPLES. III.

Multiply

1. $2a$ by $-4b$.
2. $-3b$ by $-4a$.
3. a^2 by $-a^2$.
4. $-a^3$ by $-a^2$.
5. ab by a^2b^2.
6. $-a^2$ by $-2ab$.
7. x^2y by $-xy^2$.
8. $-xy^3$ by $2x^3y$.
9. $3ab^2$ by $-4a^2b$.
10. $-2xy^4$ by $-5x^4y$.
11. $-a^3b^5$ by $-6a^5c$.
12. $-4a^3c^2$ by $-2a^4c^6$.
13. $-2ax^6y^3$ by $-3a^3x^2y^5$.
14. $3a^3bc^3x^4y$ by $-5a^2b^3cy^3$.
15. Find the values of $(-a)^2$, $(-a)^3$ and $(-a)^4$.
16. Find the values of $(-x)^2$, $(-y)^3$ and $(-z)^4$.
17. Find the values of $(-ab)^2$, $(-bc)^3$ and $(-xy)^4$.
18. Shew that the successive powers of a negative quantity are alternately positive and negative.
19. Find the squares of $3ab^2x$, $-2x^2y$ and $-5ax^2$.
20. Find the cubes of $-a^2bx^3$ and $-3ax^2y^3$.
21. Find the value of $(-a)^2 \times (-b)^2$, of $(-a)^3 \times (-b^2)^3$, and of $(-a^2b^2)^2 \times (-bc)^3$.
22. Find the value of $3abc - 4bc + 5ac^2$, when $a=4$, $b=-3$, and $c=-2$.
23. Find the value of $ab^2 + 2bc^2 + 3ca^2$, when $a=-1$, $b=-2$ and $c=-3$.
24. Find the value of $ab + 2bc + 3ca + 4ad$, when $a=1$, $b=-2$, $c=-3$ and $d=0$.

34. Division is the undoing of multiplication. In multiplication we have two factors given, and we have to

find the product. In division we have the product and one factor given, and we have to determine the other factor, so that to divide a by b is to find a quantity c such that $b \times c = a$.

As we have already remarked, the operation of division is often indicated by placing the dividend over the divisor with a line between them: thus we consider that $\frac{a}{b}$ means $a \div b$.

35. Since division is the inverse of multiplication, and multiplications can be performed in any order [Art. 31], it follows that divisions can be made in any order. Thus $a \div b \div c = a \div c \div b$.

It also follows from Art. 31 that to divide by two quantities in succession gives the same result as to divide at once by their product. Thus $a \div b \div c = a \div (bc)$.

36. Not only may a succession of divisions be performed in any order, but divisions and multiplications together may be performed in any order. For example $a \times b \div c = a \div c \times b = b \div c \times a$.

For
$$a \div c \times c = a;$$
$$\therefore\ a \div c \times c \times b = a \times b;$$
$$\therefore\ a \div c \times b \times c = a \times b;$$
$\therefore$, dividing each by c, we have
$$a \div c \times b = a \times b \div c.$$

Similarly $b \div c \times a = b \times a \div c = a \times b \div c$.

Hence we get the same result whether we divide the product of a and b by c, or divide a by c and then multiply by b, or divide b by c and then multiply by a.

If in the above we suppose $b = 1$, we have
$$a \times 1 \div c = 1 \div c \times a,$$

that is
$$a \div c = \frac{1}{c} \times a = a \times \frac{1}{c},$$
so that to divide by any quantity c is the same as to multiply by $\frac{1}{c}$.

37. Since $a^3 \times a^2 = a^5$, and $a^7 \times a^3 = a^{10}$; we have conversely $a^5 \div a^3 = a^2 = a^{5-3}$, and $a^{10} \div a^7 = a^3 = a^{10-7}$, and similarly in other cases. Hence if one power of a letter be divided by another power of the same letter, the index of the quotient is equal to the *difference* of the indices of the dividend and the divisor.

Thus
$$a^5b^2 \div a^2b = a^3b,$$
and
$$a^7b^6c^3 \div a^2b^4c = a^5b^2c^2.$$

38. We have proved [Art. 30] that,
$$a \times (-b) = -ab;$$
$$\therefore\ (-ab) \div (-b) = a, \text{ and } (-ab) \div a = -b.$$
We have also proved that
$$(-a) \times (-b) = +ab = (+a)(+b);$$
$$\therefore\ (+ab) \div (-a) = -b, \text{ and } (+ab) \div (+a) = +b.$$
Hence if the signs of the dividend and divisor are alike, the sign of the quotient is $+$; and if the signs of the dividend and divisor are unlike, the sign of the quotient is $-$: we therefore have the same **Law of Signs** in division as in multiplication.

Thus
$$-a^3b^6 \div ab^4 = -a^2b^2,$$
and
$$-2ab^3c^5 \div -3abc = \tfrac{2}{3}\, b^2c^4.$$

EXAMPLES. IV.

1. Divide $10a$ by $-5a$, $-10b$ by $4b$, and $-2x$ by $-3x$.
2. Divide a^2 by $-a$, $8ab$ by $-2b$, and $12a^2b^3$ by $-8ab^2$.
3. Divide $-6x^2y^4$ by $-3xy^2$, $-4a^2b^2$ by $3ab$, and $3x^4y^4$ by $5xy^3$.
4. Divide $abcd$ by bc, and $a^2b^3c^4d^5$ by $-ab^2c^2d^3$.
5. Divide $-2a^5b^6c^7$ by b^2c^3, $4x^6y^5$ by $-x^3y^4$, and $x^2y^2z^2$ by $-3xyz^2$.
6. Multiply $-2ab^2c^3$ by $-4a^3b^2c$ and divide the result by $a^3b^3c^3$.

39. The fundamental laws of Algebra, so far as simple expressions are concerned, are those which were marked α, β, γ, δ in the preceding articles, and which are collected below:

$$\left.\begin{aligned} +(+a) &= +a \\ +(-a) &= -a \\ -(+a) &= -a \\ -(-a) &= +a \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$\left.\begin{aligned} a \times b &= +ab \\ (-a) \times b &= -ab \\ a \times (-b) &= -ab \\ (-a) \times (-b) &= +ab \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

$$ab = ba \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\gamma),$$

$$a^m \,.\, a^n = a^{m+n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\delta).$$

It should be remarked that the laws expressed in (α), (β), (γ) have been proved to be true for *all* values of a and b; but m and n are supposed in (δ) to be positive integers.

40. We now proceed to the consideration of compound expressions.

We first observe that any compound expression can be put in the form

$$a + b + c + \&c.,$$

where a, b, c, &c. may be *any* quantities positive or negative.

For example, the expression $3x^2y - \frac{5}{2}xy^2 - 7xyz$, which by ($\alpha$) is the same as $3x^2y + (-\frac{5}{2}xy^2) + (-7xyz)$, takes the required form if we put a for $3x^2y$, b for $-\frac{5}{2}xy^2$, and c for $-7xyz$.

It therefore follows that in order to prove any theorem to be true for *any* algebraical expression, it is only necessary to prove it for the expression $a + b + c + \&c.$ where a, b, c, &c. are supposed to have any positive or negative values.

41. The sum of two or more algebraical quantities is the same in whatever order they are added.

Thus $\quad a+b+c=b+a+c=b+c+a$, &c.........(ϵ).

This follows at once from the meaning of addition, whether the quantities are positive or negative.

For example, to calculate how much a man is worth, we can take the different items of property or of debt in any order.

42. The laws (γ) and (ϵ) are together called the **Commutative Law**, which is generally enunciated in the following form: *Additions or Multiplications may be made in any order.*

43. To add the sum of two algebraical quantities gives the same result as to add the quantities *separately*. That is

$$+(a+b)=+a+b \quad \ldots\ldots\ldots\ldots\ldots\ldots(\zeta).$$

For, to put a and b together, and then to put the result to what has gone before, is the same as to put both a and b to what has gone before.

Similarly,

$$+(a+b+c+\ldots)=+a+b+c+\ldots.$$

Hence, in order to add any algebraical expression, we may add the terms *separately*.

Since the expression $a-b+c-d$ may be written in the form $a+(-b)+c+(-d)$, we have

$$\begin{aligned}+\{a-b+c-d\}&=+\{a+(-b)+c+(-d)\}\\&=+a+(-b)+c+(-d).\end{aligned}$$

When we say that we can add the *terms* of an expression separately, it must be borne in mind that the *terms include the prefixed signs.*

44. To subtract the sum of two or more algebraical quantities give the same result as to subtract the quantities *separately*. That is

$$-(a+b+c+\ldots)=-a-b-c-\ldots \quad \ldots\ldots(\eta).$$

This follows at once from the preceding, for subtraction is the undoing of addition, and hence, if to add a compound expression is the same as to add the separate terms, to subtract a compound expression must be the same as to subtract the separate terms.

45. If c be a positive integer, a and b having any values whatever, then

$$\begin{aligned}(a+b)\,c &= (a+b)+(a+b)+(a+b)+\ldots \text{ repeated } c \text{ times,}\\ &= a+b+a+b+a+b+\ldots\\ &= a+a+a+\ldots \text{ repeated } c \text{ times,}\\ &\quad + b+b+b+\ldots \text{ repeated } c \text{ times,}\\ &= ac+bc.\end{aligned}$$

Hence when c is a positive integer, we have

$$(a+b)\,c=ac+bc \quad \ldots\ldots\ldots\ldots\ldots\ldots(\theta).$$

Since division is the inverse of multiplication, it follows that when d is a positive integer

$$(a+b)\div d=a\div d+b\div d.$$

And since the operations of multiplication and division to be performed on $a+b$ can be performed on a and b *separately*, this must also be the case for the complex operation denoted by a fraction $\frac{m}{n}$.

Thus the law expressed in (θ) is true for all positive values of c; and being true for any positive value of c, it must also be true for any negative value. For, if

$$(a+b)\,c=ac+bc,$$

then $$-(a+b)\,c=-ac-bc;$$

and therefore $$(a+b)(-c)=a(-c)+b(-c).$$

Hence, for *all values* of a, b and c, we have

$$(a+b)\,c = ac + bc \quad\ldots\ldots\ldots\ldots\ldots\ldots(\theta).$$

The above is generally called the **Distributive Law.**

46. It follows conversely from (θ) that

$$(ac+bc) \div c = a + b = ac \div c + bc \div c,$$

so that the quotient obtained by dividing the sum of two algebraical quantities by a third is the sum of the quotients obtained by dividing the quantities separately by the third. This however is really included in (θ), for

$$(a+b) \div c = (a+b) \times \frac{1}{c} = a \times \frac{1}{c} + b \times \frac{1}{c} = a \div c + b \div c.$$

47. We have now considered all the fundamental laws of Algebra, which laws are true for all values, whether positive or negative, of the letters involved in their symbolical statement. In the succeeding chapters we have only to develope the consequences of these laws.

EXAMPLES. V.

1. Find the value of $3a-4b+2c$, and of $a+2b-3c$, when $a=5$, $b=-2$, and $c=-3$.

2. Find the value of $(b-c)^2+(c-a)^2+(a-b)^2$, when $a=-1$, $b=-2$, and $c=-3$.

3. Find the value of $a^3+b^3+c^3-3abc$, (i) when $a=-1$, $b=-2$ and $c=3$, and (ii) when $a=-3$, $b=-2$ and $c=5$.

4. Multiply $-2a^3bc^4$ by $-5ab^3c$, and divide the result by $6a^2b^2c^2$.

5. By what must $-5a^3bxy$ be multiplied in order that the product may be $15a^3b^2x^3y^4$?

6. Find the values of the following expressions when $a=1$, $b=2$ and $c=-3$:

(i) $\sqrt{4a^2-2bc}$, (ii) $\sqrt[3]{a+b-8c}$, (iii) $\sqrt{ab-c}$,
(iv) $a^3-3bc+2ac^2$, and (v) $(a+b-c)(a-b+c)$.

CHAPTER III.

ADDITION. SUBTRACTION. BRACKETS.

ADDITION.

48. WE have already seen [Art. 22] that any simple expression is added by writing it down, with its sign unchanged, after the expression to which it is to be added. We have also seen [Art. 43] that the terms of a compound expression may be added *separately*. We therefore have the following rule:—*To add two or more algebraical expressions, write down all the terms in succession with their signs unchanged.*

For example, the sum of $a+3b-2c$ and $d-2f-3g$ is $a+3b-2c+d-2f-3g$.

49. If some of the terms which are to be added are "like" terms, we must collect all such terms before the process is considered to be complete. Of this we have three cases, as follows:

(i) Like terms which have the same sign are added by taking the arithmetical sum of their numerical coefficients with the common sign, and affixing the common letters.

For example, to add $2a$ and $5a$ in succession gives the same result, whatever a may be, as to add $7a$; that is, $+2a+5a=+7a$. Also, to subtract $2a$ and $5a$ in succession gives the same result as to subtract $7a$; that is,

$$-2a-5a=-7a.$$

(ii) Two like terms whose signs are different are added by taking the arithmetical difference of their numerical coefficients with the sign of the greater, and affixing the common letters.

For example,

$$+5a-3a=+2a+3a-3a=+2a;$$

also $$+3a-5a=+3a-3a-2a=-2a.$$

(iii) If there are several like terms some of which are positive and some negative, the positive terms can be collected into one sum by (i), as also the negative terms: the final sum is then obtained by (ii).

For example, to add $2a+3b$ and $a-5b$.

The sum is $2a+3b+a-5b$; or, since the terms may be taken in any order [Art. 41], the sum is

$$2a+a+3b-5b=3a-2b.$$

Again, to find the sum of $2a-3b+c$, $b-2c+3d$, and $-a+3c-2d$.

The sum is $2a-3b+c+b-2c+3d-a+3c-2d$

$$=2a-a-3b+b+c-2c+3c+3d-2d=a-2b+2c+d.$$

50. It is sometimes convenient for beginners to place the different sets of "like" terms in vertical columns; so that the last example would be put down in the following form:

$$\begin{array}{rrrr}
2a & -3b & +c & \\
 & b & -2c & +3d \\
-a & & +3c & -2d \\
\hline
a & -2b & +2c & +d \\
\hline
\end{array}$$

EXAMPLES. VI.

Add together

1. $a+b$ and $a-b$.

2. $x-y$ and $-x+y$.

3. $\frac{1}{2}a+\frac{1}{2}b$ and $\frac{1}{2}a-\frac{1}{2}b$.

4. a^2-a and a^3-a^2.

5. $a+a^2+a^3$ and $2a-a^2-2a^3$.

6. $m^2+2mn+n^2$ and $m^2-2mn-2n^2$.

7. $3a^2-2ab+b^2$ and $a^2-2ab+3b^2$.

8. $a+2b-c$, $2a-b+2c$, and $-3a-b-c$.

9. $a-\frac{1}{2}b-\frac{1}{2}c$, $b-\frac{1}{2}c-\frac{1}{2}a$, and $c-\frac{1}{2}a-\frac{1}{2}b$.

10. $2a-5b+2c$, $2b-5c+2a$, and $2c-5a+2b$.

11. $4x-2y+1$, $-3x+2-y$ and $x+3y-3$.

12. $-4a-b+2$, $2+8a-5b$ and $-a-4b-2$.

13. $x^3-2x^2y-2xy^2$, $x^2y-3xy^2-y^3$, and $3xy^2-2y^3-x^3$.

14. $5a^3-2a^2b+9ab^2+17b^3$, $-2a^3+5a^2b-4ab^2-12b^3$, $b^3-4ab^2-5a^2b-a^3$, and $2a^2b-2a^3-6b^3-ab^2$.

15. Shew that, if $x=a+2b-3c$, $y=b+2c-3a$, and $z=c+2a-3b$; then will $x+y+z=0$.

16. Shew that, if $a=x-2y+3z$, $b=y-2z+3x$, and $c=z-2x+3y$; then will $a+b+c=2(x+y+z)$.

SUBTRACTION.

51. We have already seen [Art. 25] that any simple expression is subtracted by writing it down, with its sign changed, after the expression from which it is to be subtracted. We have also seen [Art. 44] that the terms of a compound expression may be subtracted *separately*. We therefore have the following rule :—

To subtract any algebraical expression, write down all its terms in succession with all their signs changed.

For example, the difference of $3a-4b+c$ and $2a+b-4c$ is

$$\begin{aligned}3a-4b+c-2a-b+4c &= 3a-2a-4b-b+c+4c\\ &= a-5b+5c.\end{aligned}$$

52. We may place the expression which is to be subtracted under that from which it is to be taken, "like" terms being for convenience placed under one another; and to obtain the required difference we must suppose all the signs of the lower line to be changed before we combine "like" terms. Thus, the example considered in Art. 51 would be written as follows:

$$\begin{array}{r} 3a-4b+c \\ 2a+b-4c \\ \hline a-5b+5c \end{array}$$

The terms of the result are obtained by combining mentally $3a$ and $-2a$, $-4b$ and $-b$, $+c$ and $+4c$.

As another example, if we have to subtract

$$3a^3-4a^2b+2ab^2-b^3$$

from $$4a^3+a^2b-ab^2,$$

the process is written as follows:

$$\begin{array}{r} 4a^3+a^2b-ab^2 \\ 3a^3-4a^2b+2ab^2-b^3 \\ \hline a^3+5a^2b-3ab^2+b^3 \end{array}$$

The above arrangement is very convenient in the division of compound algebraical expressions; but, except in division, it is best to proceed as in the example of Art. 51.

EXAMPLES. VII.

1. Subtract $3a-4b$ from $4a-3b$, and $3x-4x^2$ from $4x-3x^2$.

2. Subtract $a-2b$ from $b-2a$, and x^2-2 from $1-2x^2$.

3. Subtract $\frac{a}{2}-\frac{b}{2}$ from $\frac{a}{2}+\frac{b}{2}$, and $x-y$ from $x+y$.

4. Subtract $5x^2-4x+2$ from $3x^2+2x-1$.

5. Subtract $x-2x^2$ from x^2-2x+1.

6. Subtract $-2x^2+3xy-y^2$ from x^2+xy.

7. What is the difference between

$$3x^3+5x^2y-7xy^2+5y^3 \text{ and } -x^3+6x^2y+3xy^2-2y^3?$$

8. What is the difference between

$$3a^3+5a^2b+2b^3 \text{ and } a^3-4ab^2+5b^3?$$

9. What is the difference between

$$3a^4-5a^2b^2+2b^4 \text{ and } 3a^4-7a^2b^2-3b^4?$$

10. What must be added to $5ab-2bc+3ca$ in order that the sum may be $7ab+2ca$?

11. What must be added to $a^2+3b^2+2c^2$ in order that the sum may be b^2+c^2?

12. What must be added to $5x^3-3x^2y+2xy^2$ in order that the sum may be $x^3+x^2y+xy^2+y^3$?

Brackets.

53. To indicate that a compound expression is to be added as a whole, it is put between brackets with the + sign prefixed. But, as we have seen in Art. 48, to add any compound expression we have only to write down the terms with their signs unchanged.

Hence, when a bracket is preceded by a + sign, the bracket may be omitted. Thus

$$+(2a-b+c)=2a-b+c.$$

Hence also, any number of terms in an expression may be enclosed in brackets with the sign + placed before the bracket.

For example, we may write $a - 2b + c + 2d - 3e + f$ in the form $a + (-2b + c) + (2d - 3e + f)$.

The student should notice that the + sign is generally omitted before the *first* term of an expression within brackets. Thus $(a + b)$ is written instead of $(+a + b)$.

54. To indicate that a compound expression is to be subtracted as a whole, it is put between brackets and the − sign prefixed. But, as we have seen in Art. 51, to subtract any compound expression we have only to write down the terms in succession with all their signs changed.

Hence, when a bracket is preceded by the − sign, the brackets may be omitted, provided that the signs of all the terms within the brackets are changed.

Thus $$-(2a - b + c) = -2a + b - c.$$

Hence also, any number of terms in an expression may be enclosed in brackets with the sign − placed before the bracket, provided that we change the signs of all the terms which are placed in the brackets.

Thus $a - 2b + c + 2d - 3e + f$ may be written in the following form: $a - (2b - c) - (-2d + 3e - f)$.

55. Sometimes brackets are put within brackets: in this case the different pairs of brackets are taken of different shapes to prevent confusion.

Thus $a - [b + \{c - (d + e)\}]$; which means that we are to add to b the whole quantity within the brackets marked $\{\ \}$, and then subtract the result from a; and, to find the quantity within the brackets marked $\{\ \}$, we must add d and e, and then subtract the sum from c.

When there are several pairs of brackets they may be removed *one at a time* by the rules of Arts. 53 and 54; and it is best for beginners to remove at every stage the *innermost* bracket.

Thus
$$\begin{aligned} &a-[b+\{c-(d+e)\}] \\ =\ &a-[b+\{c-d-e\}] \\ =\ &a-[b+c-d-e] \\ =\ &a-b-c+d+e. \end{aligned}$$

EXAMPLES. VIII.

Simplify the following expressions by removing the brackets and collecting like terms:

1. $x-\{y-(z-x)\}$.
2. $a-[a-\{a-(a-a)\}]$.
3. $1-[2-\{3-(4-5)\}]$.
4. $a+b-[a-b+\{a+b-(a-b)\}]$.
5. $5-[4+\{5-(4+\overline{5-4})\}]$.
6. $x-[y-\{z-(x-\overline{y-z})\}]$.
7. $3x-\{2y+(5z-\overline{3x+y})\}$.
8. $\{2x-(5y-\overline{3z+7})\}-[4+\{x-(3y+2z+5)\}]$.
9. $[2a-\{3b+(4c-\overline{3b+2a})\}]$.
10. $x-(y-z)+\{2z-\overline{3y-5x}\}$.
11. $a-2b-\{3a-(b-c)-5c\}$.
12. $a-[3b+\{3c-(d-b)+a\}-2a]$.

CHAPTER IV.

MULTIPLICATION.

56. **WE** have already seen how to multiply two simple algebraical expressions: we now proceed to consider the multiplication of compound expressions.

57. We know [Art. 45] that the product of the sum of *any* two algebraical quantities by a third is equal to the sum of the products obtained by multiplying the quantities separately by the third.

Hence $\{a+(b+c)\}\,x = ax + (b+c)\,x;$

$$\therefore\ (a+b+c)\,x = ax + bx + cx.$$

And similarly,

$$(a+b+c+d+\ldots)\,x = ax + bx + cx + dx + \ldots.$$

Thus *the product of any compound expression and a simple one is the sum of the products obtained by multiplying the separate terms by the simple expression.*

58. We will now consider the most general case of multiplication, namely the multiplication of any two compound expressions. We have to find

$$(a+b+c+\ldots)\times(x+y+z+\ldots).$$

Put M for $x+y+z+\dots$; then, by the last Article, we have

$$\begin{aligned}(a+b+c+\dots)M &= aM+bM+cM+\dots\\ &= Ma+Mb+Mc+\dots\ \text{[Art. 31]}.\end{aligned}$$

$$=(x+y+z+\dots)a+(x+y+z+\dots)b+(x+y+z+\dots)c+\dots$$

$$=ax+ay+az+\dots+bx+by+bz+\dots+cx+cy+cz+\dots+\dots.$$

Hence $\quad (a+b+c+\dots)(x+y+z+\dots)$

$$=ax+ay+az+\dots+bx+by+bz+\dots+cx+cy+cz+\dots.$$

Thus *the product of any two compound expressions is the sum of the products obtained by multiplying every term of the multiplicand by every term of the multiplier.*

For example,

$$(a+b)(c+d)=ac+bc+ad+bd.$$

Again,

$$\begin{aligned}(2a+5b)(3a+2b) &= 2a\times 3a+5b\times 3a+2a\times 2b+5b\times 2b\\ &= 6a^2+15ab+4ab+10b^2=6a^2+19ab+10b^2.\end{aligned}$$

As another example let us find $(a-b)\times(c-d)$. We must first write this in the form $\{a+(-b)\}\{c+(-d)\}$, and we then have for the product

$$ac+(-b)c+a(-d)+(-b)(-d),$$

which by Art. 30 is equal to

$$ac-bc-ad+bd.$$

From the last example we see that the rule we have given above for the multiplication of two algebraical expressions will apply in all cases, since *terms* are supposed to *include the prefixed signs.*

59. The following are important examples:—

(i) $\quad (a+b)^2=(a+b)(a+b)=aa+ba+ab+bb;$

$$\therefore\ (a+b)^2=a^2+2ab+b^2.$$

Hence, *the square of the sum of any two quantities is equal to the sum of their squares plus twice their product.*

(ii) $$(a-b)^2=(a-b)(a-b)=aa+(-b)a+a(-b)+(-b)(-b)=a^2-ab-ab+b^2;$$

$$\therefore\ (a-b)^2=a^2-2ab+b^2.$$

Hence, *the square of the difference of any two quantities is equal to the sum of their squares minus twice their product.*

(iii) $$(a+b)(a-b)=aa+ba+a(-b)+b(-b)=a^2+ab-ab-b^2;$$

$$\therefore\ (a+b)(a-b)=a^2-b^2.$$

Hence, *the product of the sum and difference of any two quantities is equal to the difference of their squares.*

60. It is usual to exhibit the process of multiplication in the following convenient form.

$$\begin{array}{lllll}
a^2 & +2ab & -b^2 & & \\
a^2 & -2ab & +b^2 & & \\
\hline
a^4 & +2a^3b & -a^2b^2 & & \\
 & -2a^3b & -4a^2b^2 & +2ab^3 & \\
 & & a^2b^2 & +2ab^3 & -b^4 \\
\hline
a^4 & & -4a^2b^2 & +4ab^3 & -b^4
\end{array}$$

The multiplier is placed under the multiplicand and a line is drawn. The successive terms of the multiplicand, namely a^2, $+2ab$ and $-b^2$, are multiplied by a^2, the first term on the left of the multiplier; and the products a^4, $+2a^3b$ and $-a^2b^2$ which are thus obtained are put in a horizontal row. The terms of the multiplicand are then multiplied by $-2ab$, the second term of the multiplier, and the products thus obtained are put in another horizontal row, the terms being so placed that 'like' terms are under one another. The terms of the multiplicand are then multiplied by b^2, the last term of the multiplier, and the products thus obtained are put in a third horizontal row, 'like' terms being again placed under one another. The final result is then obtained by adding the three rows of partial products; and this final sum can be readily written down, since the different sets of 'like' terms are in vertical columns.

61. The following are additional examples of multiplications arranged as in the preceding article.

$$\begin{array}{lll} a+b & & \\ a+b & & \\ \hline a^2 & +ab & \\ & +ab & +b^2 \\ \hline a^2 & +2ab & +b^2 \end{array} \qquad \begin{array}{lll} a-b & & \\ a-b & & \\ \hline a^2 & -ab & \\ & -ab & +b^2 \\ \hline a^2 & -2ab & +b^2 \end{array} \qquad \begin{array}{lll} a+b & & \\ a-b & & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

$$\begin{array}{lll} a+\sqrt{2}b & & \\ a-\sqrt{2}b & & \\ \hline a^2 & +\sqrt{2}ba & \\ & -\sqrt{2}ba & -2b^2 \\ \hline a^2 & & -2b^2 \end{array} \qquad \begin{array}{llllll} a+b+c & & & & & \\ a+b+c & & & & & \\ \hline a^2 & +ab & +ac & & & \\ & +ab & & +b^2 & +bc & \\ & & +ac & & +bc & +c^2 \\ \hline a^2 & +2ab & +2ac & +b^2 & +2bc & +c^2 \end{array}$$

62. If in any expression consisting of several terms which contain different powers of the same letter, the term which contains the highest power of that letter be put first on the left, the term which contains the next highest power be put next, and so on; the terms, if any, which do not contain the letter being put last; then the whole expression is said to be *arranged according to descending powers* of that letter. Thus, the expression $a^3+a^2b+ab^2+b^3$, is arranged according to descending powers of the letter a. In like

manner we say that the above expression is arranged according to *ascending* powers of the letter b.

63. Although it is not *necessary* to arrange the terms either of the multiplicand or of the multiplier in any particular order, it will be found convenient to arrange *both* expressions either according to descending or according to ascending powers of the same letter: some trouble in the arrangement of the different sets of like terms of the product in vertical columns will thus be avoided.

EXAMPLES IX.

1. Multiply $x+2y$ by $x-2y$.
2. Multiply $a+\frac{1}{2}b$ by $a-\frac{1}{2}b$.
3. Multiply $1+ax+a^2x^2$ by $1-ax$.
4. Multiply y^2+by+b^2 by $b-y$.
5. Multiply x^2-3ax by $x+3a$.
6. Multiply y^2+2by by $y-2b$.
7. Multiply $x+y-z$ by $x-y+z$.
8. Multiply $4a+2b-3c$ by $4a-2b+3c$.
9. Multiply a^2+a+1 by a^2-a+1.
10. Multiply $a^2+\sqrt{2}a+1$ by $a^2-\sqrt{2}a+1$.
11. Multiply x^4+x^2+1 by x^4-x^2+1.
12. Multiply $x^2+\sqrt{3}x+1$ by $x^2-\sqrt{3}x+1$.
13. Multiply $3x^2-xy-2y^2$ by $4y^2+2xy+x^2$.
14. Multiply $x^2-2xy-y^2$ by $y^2+2xy-x^2$.

64. A term which is the product of n letters is said to be of n *dimensions*, or of the nth *degree*. Numerical factors are not to be counted in reckoning the number of dimensions. Thus abc is of three dimensions, or of the

third degree; and $5a^2b^2c$, that is $5aabbc$, is of five dimensions, or of the fifth degree.

The *degree of an expression* is the degree of that term which is of highest dimensions.

In estimating the degree of a term, or of an expression, we sometimes take into account only a particular letter, or particular letters: thus we say that ax^2+bx+c is of the second degree *in* x, and that ax^2y+bxy^2 is of the third degree *in* x *and* y.

When all the terms of an expression are of the same dimensions, the expression is said to be *homogeneous*. Thus $a^3+3a^2b-5b^3$ is a homogeneous expression, every term being of three dimensions.

65. *The product of any two homogeneous expressions is homogeneous.* For the different terms of the product are obtained by multiplying any term of the multiplicand by any term of the multiplier, and the number of dimensions in the product of any two simple quantities is clearly the sum of the number of dimensions in the separate quantities; hence if all the terms of the multiplicand are of the same degree, as also all the terms of the multiplier, it follows that all the terms of the product are of the same degree.

The fact that two expressions which are to be multiplied are homogeneous should in all cases be noticed by a student; for if the product obtained is not homogeneous, it is at once seen that there is an error.

66. We now return to the three important cases of multiplication considered in Art. 59, namely

$$(a+b)^2=a^2+2ab+b^2 \quad\text{..................(i)},$$

$$(a-b)^2=a^2-2ab+b^2 \quad\text{..................(ii)},$$

$$(a+b)(a-b)=a^2-b^2 \quad\text{........................(iii)}.$$

Since the laws from which the above results were deduced were proved to be true for all algebraical quantities whatever, we may substitute for a and for b any other algebraical quantities, or algebraical expressions, without affecting the truth of the results.

In the first place, let us put $-b$ in the place of b in (i); we then have

$$\{a+(-b)\}^2 = a^2 + 2a(-b) + (-b)^2,$$

that is, $$(a-b)^2 = a^2 - 2ab + b^2.$$

Thus (ii) is seen to be really included in (i).

Now put $b+c$ in the place of b in (i); we then have

$$\{a+(b+c)\}^2 = a^2 + 2a(b+c) + (b+c)^2;$$

$$\therefore \{a+b+c\}^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2.$$

Thus $$\{a+b+c\}^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \ldots \text{(iv)}.$$

Now put $-c$ for c in (iv), and we have

$$\{a+b+(-c)\}^2 = a^2 + b^2 + (-c)^2 + 2ab + 2a(-c) + 2b(-c);$$

$$\therefore (a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$$

Again put $b+c$ in the place of b in (iii); we then have

$$\{a+(b+c)\}\{a-(b+c)\} = a^2 - (b+c)^2.$$

We can similarly, by assuming the truth of (iii), write down at once the following products

$$(a^2+b^2)(a^2-b^2) = (a^2)^2 - (b^2)^2 = a^4 - b^4,$$

$$(a^2+\sqrt{2}b)(a^2-\sqrt{2}b) = a^4 - 2b^2,$$

$$\begin{aligned}(a-b+c)(a+b-c) &= \{a-(b-c)\}\{a+(b-c)\}\\ &= a^2 - (b-c)^2 = a^2 - b^2 + 2bc - c^2,\end{aligned}$$

$$\begin{aligned}(a^2+ab+b^2)(a^2-ab+b^2) &= (\overline{a^2+b^2}+ab)(\overline{a^2+b^2}-ab)\\ &= (a^2+b^2)^2 - a^2b^2 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = a^4 + a^2b^2 + b^4.\end{aligned}$$

67. We have found in the preceding Article, and also in Art. 61, the square of the sum of *three* algebraical quantities; and we can by the same methods obtain the square of the sum of more than three quantities. The square of the sum of any number of algebraical quantities may also be found in the following manner.

Suppose we wish to find

$$(a+b+c+d+\ldots)(a+b+c+d+\ldots).$$

We know that the product of any two algebraical expressions is equal to the sum of the partial products obtained by multiplying every term of the multiplicand by every term of the multiplier. If we multiply the term a of the multiplicand by the term a of the multiplier, we obtain the term a^2 of the product: we similarly obtain the terms b^2, c^2, &c. We can multiply any term, say b, of the multiplicand by any different term, say d, of the multiplier; and we thus obtain the term bd of the product. But we also obtain the term bd of the product by multiplying the term d of the multiplicand by the term b of the multiplier, and we can obtain the term bd in no other way, so that every such term as bd, in which the letters are different, occurs *twice* in the product. The required product is therefore the sum of the squares of all the quantities a, b, c, d &c., together with twice the product of every pair.

Hence we have the following important proposition:—*The square of the sum of any number of algebraical quantities is equal to the sum of their squares together with twice the product of every pair.* For example, to find the square of $(a+b+c)$. The squares of the separate terms are a^2, b^2, c^2.

The products of the different pairs of terms are ab, ac and bc. Hence, from the above rule, we have

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc.$$

Similarly,

$$\begin{aligned}(a+2b-3c)^2 &= a^2+(2b)^2+(-3c)^2+2a(2b)\\ &\qquad +2a(-3c)+2(2b)(-3c)\\ &= a^2+4b^2+9c^2+4ab-6ac-12bc.\end{aligned}$$

EXAMPLES X.

Ex. 1. Write down the squares of the following expressions:—

(i) $2a+3b$, (ii) a^2-5ab, (iii) $-3xy+y^2$, (iv) $a-2b+3c$, (v) a^2-ab+b^2, (vi) x^2+x+1, (vii) x^2-xy+y^2, (viii) $bc+ca+ab$, (ix) $a+b+c+d$, (x) $a-b+c-d$, (xi) x^3+x^2+x+1, and (xii) x^3-x^2+x-1.

Ex. 2. Multiply (i) x^2+xy+y^2 by x^2-xy+y^2, (ii) $3x^2-xy+2y^2$ by $3x^2+xy-2y^2$, (iii) $x+y-z$ by $y+z-x$, and (iv) $a^3+a^2b+ab^2+b^3$ by $a^3-a^2b+ab^2-b^3$.

68. We will now find the continued product of $x+a$, $x+b$ and $x+c$. The process is written as under:—

$$\begin{array}{llll}
x+a & & & \\
x+b & & & \\
\hline
x^2+ax & & & \\
\quad +bx+ab & & & \\
\hline
x^2+(a+b)x+ab & & & \\
x+c & & & \\
\hline
x^3+(a+b)x^2+abx & & & \\
\quad +cx^2 & +c(a+b)x+abc & & \\
\hline
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc & & &
\end{array}$$

We have arranged the result in a way which is frequently required, namely according to descending powers of x; and all the terms which contain the same powers of x are collected together.

69. If in the continued product found in Art. 68, we suppose b and c each equal to a, we obtain

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \;\ldots\ldots\ldots\ldots\text{(i)}.$$

This result would of course be obtained by multiplying the known value of $(x+a)^2$ by $x+a$.

If we put $-a$ instead of a in (i), it will become

$$\{x+(-a)\}^3 = x^3 + 3(-a)x^2 + 3(-a)^2x + (-a)^3;$$

$$\therefore\; (x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \;\ldots\ldots\ldots\ldots\text{(ii)}.$$

The results (i) and (ii) should be remembered.

We shall in a succeeding chapter shew how to write down any power of a binomial expression, by a rule which is called the BINOMIAL THEOREM; for the present it is sufficient to know how to write down the *cube* of a binomial.

EXAMPLES XI.

1. Multiply $3a^2+ab-b^2$ by $a^2-2ab-3b^2$.
2. Multiply $x^2-2xy-y^2$ by $y^2-2xy-x^2$.
3. Multiply $\frac{3}{2}x^2-x+\frac{2}{3}$ by $3x^2+2x+\frac{4}{3}$.
4. Multiply $a^2+b^2+c^2-bc-ca-ab$ by $a+b+c$.
5. Multiply $4a^2+9b^2+c^2-3bc-2ca-6ab$ by $2a+3b+c$.
6. Multiply $ax+a^2x^2+a^3x^3$ by $ax-1$.
7. Multiply $a+b$, a^2+b^2 and $a-b$.
8. Multiply x^2+y^2, x^4+y^4 and x^2-y^2.
9. Multiply x^2-xy+y^2, x^2+xy+y^2 and $x^4-x^2y^2+y^4$.
10. Multiply a^2+a+1, a^2-a+1 and a^4-a^2+1.

11. Multiply $a+b-c$, $b+c-a$, $c+a-b$ and $a+b+c$.

12. Multiply $x+2y-3z$, $2y+3z-x$ and $3z+x-2y$.

13. Multiply $(x-a)^2$, $(x+a)^2$ and $(x^2+a^2)^2$.

14. Multiply $x^2+\sqrt{2}xy+y^2$ by $x^2-\sqrt{2}xy+y^2$.

15. Multiply $a^2+\sqrt{3}ab+b^2$ by $a^2-\sqrt{3}ab+b^2$.

16. Find the cubes of $2a+3b$ and $3a-2b$.

17. Find the cubes of $a+b+c$ and $a+b-c$.

18. In the expression x^3-2x^2+3x-4 substitute $a-2$ for x, and arrange the result according to descending powers of a.

19. Shew that

(i) $(2x+1)^2+(x-1)^2=4x^2+(x+1)^2+1$.

(ii) $(2x+1)^2+(x+2)^2=(x-2)^2+4x(x+3)+1$.

(iii) $(x^2+x+1)^2+(x^2-x+1)^2=2(x^4+3x^2+1)$.

20. Shew that

$$a^3-b^3=(a-b)(a^2+ab+b^2)=(a-b)^3+3ab(a-b)$$
$$=(a+b)^3-3ab(a+b)-2b^3.$$

21. Shew that

$$(x+y)(x+z)+(y+z)(y+x)+(z+x)(z+y)-(x+y+z)^2=yz+zx+xy.$$

22. Shew that

$$(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)=0.$$

CHAPTER V.

DIVISION.

70. WE have already considered the division of one simple expression by another. We have also seen (Art. 46) that the quotient obtained by dividing the sum of two algebraical quantities by a third is the sum of the quotients obtained by dividing the quantities separately by the third; and we can shew by the method of Art. 57 that when any compound expression is divided by a simple one the quotient is the sum of the quotients obtained by dividing the separate terms of the compound expression by the simple one.

Thus $(a^2x^2 - 3ax) \div ax = a^2x^2 \div ax - 3ax \div ax = ax - 3.$

And $(12x^3 - 5ax^2 - 2a^2x) \div 3x$

$$= 12x^3 \div 3x - 5ax^2 \div 3x - 2a^2x \div 3x = 4x^2 - \tfrac{5}{3}ax - \tfrac{2}{3}a^2.$$

The above may also be written as follows:

$$\frac{a^2x^2 - 3ax}{ax} = \frac{a^2x^2}{ax} - \frac{3ax}{ax} = ax - 3,$$

and
$$\frac{12x^3 - 5ax^2 - 2a^2x}{3x} = \frac{12x^3}{3x} - \frac{5ax^2}{3x} - \frac{2a^2x}{3x}$$

$$= 4x^2 - \tfrac{5}{3}ax - \tfrac{2}{3}a^2.$$

71. We must now consider the most general case of division, namely the division of one compound expression by another.

Having arranged both the dividend and the divisor according to descending powers of some common letter, a suppose, we proceed as follows.

Divide the term which contains the highest power of a in the dividend by the term which contains the highest power of a in the divisor; then multiply the whole divisor by the quantity so found, and subtract the product from the dividend.

Now treat the remainder as a new dividend and go on repeating the process, until there is no remainder: we shall then have subtracted from the dividend a series of products of the divisor which are together equal to the dividend, that is, we shall have found the expression which when multiplied by the divisor is equal to the dividend, and that is what we require.

The process above described will be made clearer by the following example. The arrangement is the same as in Arithmetic.

Divide $8a^3 + 8a^2b + 4ab^2 + b^3$ by $2a + b$.

$$\begin{array}{l}
2a + b\,)\,8a^3 + 8a^2b + 4ab^2 + b^3\,(\,4a^2 + 2ab + b^2 \\
\qquad\quad\;\; \underline{8a^3 + 4a^2b\qquad\quad} \\
\qquad\qquad\qquad + 4a^2b + 4ab^2 + b^3 \\
\qquad\qquad\qquad \underline{+ 4a^2b + 2ab^2\quad\;\;} \\
\qquad\qquad\qquad\qquad\quad + 2ab^2 + b^3 \\
\qquad\qquad\qquad\qquad\quad \underline{+ 2ab^2 + b^3}
\end{array}$$

We have first to divide the term which contains the highest power of a in the dividend by the term which contains the highest power of a in the divisor; that is, since the dividend and the divisor are both arranged according to descending powers of a, we have to divide the *first* term of the dividend by the *first* term of the divisor: the result is $4a^2$, which is placed as the first term of the quotient. We then multiply the whole divisor by $4a^2$, placing the terms of the product under the 'like' terms of the dividend, and subtract from the dividend; we thus obtain the remainder $4a^2b + 4ab^2 + b^3$. We divide the first term of the remainder by the first term of the divisor; we thus obtain $+ 2ab$, which is placed in the quotient. We then multiply the whole divisor by $+ 2ab$, placing the terms of the product under 'like' terms of the remainder just found, and subtract from that remainder; we thus obtain the remainder $2ab^2 + b^3$. We divide the first term of the last remainder by the first term of the divisor; we thus obtain $+ b^2$, which we place in the quotient. We then multiply the whole divisor by $+ b^2$, placing the terms of the product under 'like' terms of the last remainder; we subtract as before, and there is no remainder. Since there is no remainder after the last subtraction, the dividend must be equal to the sum of the different quantities which have been subtracted from it; but we have subtracted in succession the divisor multiplied by $4a^2$, by $+ 2ab$, and by $+ b^2$; we have therefore subtracted altogether the divisor multiplied by $(4a^2 + 2ab + b^2)$. And, since the divisor multiplied by $4a^2 + 2ab + b^2$ is equal to the dividend, the required quotient is $4a^2 + 2ab + b^2$.

The dividend and divisor may be arranged according to *ascending* instead of according to *descending* powers of the common letter; in fact, in the example just considered the dividend and divisor are arranged according to ascending powers of b.

The following are additional examples :—

$$
\begin{array}{rllllll}
a^2+b^2\,) & a^4 & -a^3b & +2a^2b^2 & -ab^3 & +b^4 & (\,a^2-ab+b^2 \\
 & a^4 & & +a^2b^2 & & & \\
\cline{2-6}
 & & -a^3b & +a^2b^2 & -ab^3 & +b^4 & \\
 & & -a^3b & & -ab^3 & & \\
\cline{3-6}
 & & & +a^2b^2 & & +b^4 & \\
 & & & +a^2b^2 & & +b^4 & \\
\cline{4-6}
\end{array}
$$

$$
\begin{array}{rllllll}
a^2-ab+b^2\,) & a^4 & & +a^2b^2 & & +b^4 & (\,a^2+ab+b^2 \\
 & a^4 & -a^3b & +a^2b^2 & & & \\
\cline{2-6}
 & & a^3b & & & +b^4 & \\
 & & a^3b & -a^2b^2 & +ab^3 & & \\
\cline{3-6}
 & & & a^2b^2 & -ab^3 & +b^4 & \\
 & & & a^2b^2 & -ab^3 & +b^4 & \\
\cline{4-6}
\end{array}
$$

In the last example we have placed the terms of the dividend apart, in order that 'like' terms may be placed under one another without altering the order of the terms in descending powers of a.

72. A general result expressed symbolically is called a *formula*.

The following formulae can easily be verified, and should be remembered:

(i) $(x^2+2ax+a^2)\div(x+a)=x+a,$

(ii) $(x^2-2ax+a^2)\div(x-a)=x-a,$

(iii) $(x^2-a^2)\div(x-a)=x+a,$

(iv) $(x^3-a^3)\div(x-a)=x^2+ax+a^2,$

(v) $(x^3+a^3)\div(x+a)=x^2-ax+a^2.$

73. The following are important examples.

Ex. 1. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$
\begin{array}{l}
a+b+c\,)\,a^3-3abc+b^3+c^3\,(\,a^2-ab-ac+b^2-bc+c^2 \\
\qquad\quad \underline{a^3+a^2b+a^2c} \\
\qquad\qquad -a^2b-a^2c-3abc+b^3+c^3 \\
\qquad\qquad \underline{-a^2b-ab^2-abc} \\
\qquad\qquad\qquad -a^2c+ab^2-2abc+b^3+c^3 \\
\qquad\qquad\qquad \underline{-a^2c-abc-ac^2} \\
\qquad\qquad\qquad\qquad +ab^2-abc+ac^2+b^3+c^3 \\
\qquad\qquad\qquad\qquad \underline{+ab^2+b^3+b^2c} \\
\qquad\qquad\qquad\qquad\qquad -abc+ac^2-b^2c+c^3 \\
\qquad\qquad\qquad\qquad\qquad \underline{-abc-b^2c-bc^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad +ac^2+bc^2+c^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{+ac^2+bc^2+c^3}
\end{array}
$$

The student will see that in the above we have made no attempt to put 'like' terms under one another, as the subtractions can be easily performed without that help. Where, as in the above example, more than two letters are involved, it is not sufficient to arrange the terms according to descending powers of a, we give to b the precedence over c.

Ex. 2. Divide $x^3-(a+b+c)x^2+(bc+ca+ab)x-abc$ by $x-a$.

$$
\begin{array}{l}
x-a\,)\,x^3-(a+b+c)x^2+(bc+ca+ab)x-abc\,(\,x^2-(b+c)x+bc \\
\qquad\; \underline{x^3-ax^2} \\
\qquad\quad -(b+c)x^2+(bc+ca+ab)x-abc \\
\qquad\quad \underline{-(b+c)x^2+a(b+c)x} \\
\qquad\qquad\qquad\qquad\qquad bcx-abc \\
\qquad\qquad\qquad\qquad\qquad \underline{bcx-abc}
\end{array}
$$

74. In the process of division as described in Art. 71, the dividend and divisor being arranged according to descending powers of x, we subtract from the dividend at the first subtraction an expression such that the term in it which contains the highest power of x is the same as the term in the dividend which contains the highest power of x; hence it follows that the highest power of x contained in the remainder after the first subtraction will be *less* than the highest power of x contained in the dividend. Similarly, the highest power of x contained in every remainder will be less than in the preceding remainder; and hence by proceeding far enough we must come to a stage where there is no remainder, or else where there is a remainder the highest power of x in which is less than the highest power of x in the divisor, and in this latter case the division cannot be exactly performed.

For example, if we divide $x^2+3ax+a^2$ by $x+a$, we have

$$
\begin{array}{rl}
x+a\,) & x^2+3ax+a^2\,(\,x+2a \\
 & \underline{x^2+ax} \\
 & \phantom{x^2+{}}2ax+a^2 \\
 & \phantom{x^2+{}}\underline{2ax+2a^2} \\
 & \phantom{x^2+2ax+{}}\underline{-\,a^2}
\end{array}
$$

Thus $(x^2+3ax+a^2)\div(x+a)=x+2a$, with remainder $-a^2$.

We may express the above in the following form:

$$x^2+3ax+a^2=(x+2a)(x+a)-a^2.$$

We have also, by arranging the dividend and the divisor differently,

$$\begin{array}{r@{\,}l}
a+x\,) & a^2+3ax+x^2\,(\,a+2x \\
 & \underline{a^2+\;\;ax\qquad\;\;} \\
 & \qquad 2ax+\;\;x^2 \\
 & \qquad \underline{2ax+2x^2} \\
 & \qquad\qquad \underline{-\;\;x^2}
\end{array}$$

Hence, when there is a remainder, a change in the order of the dividend and the divisor leads to a result of different form. This is what might be expected considering that in the first case we have found what the divisor must be multiplied by in order to agree with the dividend so far as the terms which contain x are concerned; and in the second case we have found what the divisor must be multiplied by, in order to agree with the dividend so far as the terms which contain a are concerned.

When we are told to divide one expression by another, it is always understood that if there is a remainder, the remainder is either not to contain a particular letter, or to contain the lowest possible power of that letter.

EXAMPLES XII.

1. Divide $5x^4-7x^3y+4x^2y^2$ by $4x^2$.
2. Divide x^2-4y^2 by $x-2y$.
3. Divide $x^2-\frac{1}{4}y^2$ by $x+\frac{1}{2}y$.
4. Divide $1-a^3x^3$ by $1-ax$.
5. Divide x^3-y^3 by $y-x$.
6. Divide $3x^2-4x-4$ by $2-x$.
7. Divide x^3-8x-3 by $3-x$.
8. Divide x^4+x^2+1 by x^2+x+1.
9. Divide x^8+x^4+1 by x^4+x^2+1.
10. Divide x^4+4x^2+16 by x^2+2x+4.

11. Divide $x^2-y^2-z^2+2yz$ by $y-z-x$.

12. Divide $x^2-4y^2-z^2-4yz$ by $2y+z-x$.

13. Divide $y^2z^2-z^2x^2-x^2y^2+2x^2zy$ by $yz-zx+xy$.

14. Divide $a^3b^3-c^3d^3$ by $ab-cd$.

15. Divide a^6x^6-1 by a^2x^2+ax+1.

16. Divide $a^2+4b^2+9c^2-12bc-6ca+4ab$ by $3c-2b-a$.

17. Divide x^4+2x^2+x+2 by x^2+x+1.

18. Divide x^4+2x^2-x+2 by x^2-x+1.

19. Divide x^6-3x^3-40 by $2-x$.

20. Divide $x^3+ax^2-3a^2x-6a^3$ by $x-2a$.

21. Divide $1-x+5x^3-3x^4$ by $1-2x+3x^2$.

22. Divide $x^3+y^3-z^3+3xyz$ by $x+y-z$.

23. Divide $8x^3-y^3+z^3+6xyz$ by $y-z-2x$.

24. Divide $x^6+2x^5-4x^4-2x^3+12x^2-2x-1$ by x^2+2x-1.

25. Divide $1-5x^4+4x^5$ by $(1-x)^2$.

26. Divide $a^3+3a^2b+3ab^2+b^3+c^3$ by $a+b+c$.

27. Divide $3x^5-12x^4+17x^3-9x^2+1$ by $(1-x)^3$.

28. Divide $35+4x-16x^2+19x^3-6x^4$ by $7+5x-3x^2$.

29. Divide $x^3+x^2(y+z)-xyz-y^2z-yz^2$ by x^2-yz.

30. Divide $ax^3+(2a^2+ac-b^2)x^2+(2a^2c-2ab^2-b^2c)x-2ab^2c$ by $x+2a$.

MISCELLANEOUS EXAMPLES. I.

A.

1. Find the value of $(y-z)^2+(z-x)^2+(x-y)^2$ when $x=-1$, $y=0$ and $z=1$.

2. Add together $3a^2-2ca-2ab$, $2b^2+3bc+3ba$ and $c^2-2ac-2bc$.

3. Shew that

$$a^3+b^3+c^3-3abc-a(a^2-bc)-b(b^2-ca)-c(c^2-ab)=0.$$

4. Multiply $2x^2-\frac{1}{2}x+\frac{1}{3}$ by $\frac{1}{2}x+3$,
and $\frac{1}{4}x^2+\frac{1}{2}xy+y^2$ by $\frac{1}{4}x^2-\frac{1}{2}xy+y^2$.

5. Simplify $(x+1)(x+2)(x+3)-(x-1)(x-2)(x-3)$.

6. Divide $2-12x^5+10x^6$ by $1-2x+x^2$.

B.

1. Find the value of $c^2 + b(c+a)$, and of $\sqrt{2bc} - a$, when $a=11$, $b=3$ and $c=6$.

2. Subtract $5a^4 - 3a^3b + 4a^2b^2$ from $5b^4 - 3b^3a + 4b^2a^2$.

3. Simplify $x - (1 - \overline{1 - x})$, $3x - \overline{7 - 4x}$
and $b - 2a - \{c - a - (b - a + c)\}$.

4. Multiply $9x^2 - 1$ by $x^2 + \frac{1}{9}$, and $a + b + c$ by $a + b - c$.

5. Shew that

$$2(a^2 + b^2 + c^2 - bc - ca - ab) = (b-c)^2 + (c-a)^2 + (a-b)^2.$$

6. Divide $x^3 - 3x^2 + 3x + y^3 - 1$ by $x + y - 1$.

C.

1. Find the numerical value of

$$\frac{\sqrt{a^2 + 2bc}}{a} + \frac{\sqrt{b^2 + ca}}{b} + \frac{\sqrt{c^2 + ab}}{c},$$

when $a=4$, $b=3$ and $c=-2$.

2. Add together $3a^2b - 5ab^2 + 7b^3$ and $2a^3 - \frac{1}{2}a^2b + 5ab^2 - \frac{4}{5}b^3$.

3. Multiply $ax^2 - x + a$ by $ax^2 + x + a$, and find the square of $2a + b - 3c$.

4. Simplify $(x+y)^2 - (x+y)(x-y) - \{x(2y-x) - y(2x-y)\}$.

5. The product of two algebraical expressions is

$$x^6 + x^5y + x^4y^2 - x^3y^3 + y^6,$$

and one of them is $x^2 + xy + y^2$; what is the other?

6. Prove

(i) $a(b-c) + b(c-a) + c(a-b) = 0$,

(ii) $a^2(b-c) + b^2(c-a) + c^2(a-b) + (b-c)(c-a)(a-b) = 0$.

D.

1. If $a=1$, $b=2$ and $c=-3$, find the value of $6a + 3b + 4c$, and of $a^3 + b^3 + c^3 - 3abc$.

2. Simplify $3\{x - 2(y-z)\} - [4y + \{2y - (z-x)\}]$.

3. What must be added to $(a+b+c)^2$ that the sum may be $(a-b-c)^2$?

4. Multiply a^3+b^3 by $a-b$, and divide the result by $a+b$. Find the continued product of

$$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8).$$

5. Divide x^3+z^3 by $x+z$, and from the result write down the quotient when $(x+y)^3+z^3$ is divided by $x+y+z$.

6. Prove

(i) $(x+y)(x-y)+(y+z)(y-z)+(z+x)(z-x)=0$,

(ii) $(x-y)^2+(y-z)^2+(z-x)^2$
$$=2(x-y)(x-z)+2(y-z)(y-x)+2(z-x)(z-y).$$

E.

1. Find the value of

$$\frac{2\sqrt{(a^2+b^2)}+\sqrt[3]{(a^2+b^2-c^2+6d^2)}}{d-c+b-a},$$

when $a=3$, $b=4$, $c=5$ and $d=6$.

2. Simplify $4x-(y-x)-3\{2y-3(x+y)\}$ and

$$2x-3y-4(x-2y)+5\{3x-2(x-y)\},$$

and multiply the two results.

3. Shew that $(a+b)^3=a^3+b^3+3ab(a+b)$, and verify the result when $a=1$ and $b=-2$.

4. Find the coefficient of x^2 in the product of $x^2-(a-b)x-ab$ and $x^2+(a+b)x+ab$.

5. Divide the difference of $(2a+3b)^2$ and $(3a+2b)^2$ by $a+b$; divide also the sum of $(2a-3b)^3$ and $(3a-2b)^3$ by $a-b$.

6. Shew that

$$(a+b+c-d)(a+b-c+d)+(a-b+c+d)(-a+b+c+d)=4(ab+cd),$$

and that

$$a^3(b-c)+b^3(c-a)+c^3(a-b)+(b-c)(c-a)(a-b)(a+b+c)=0.$$

F.

1. Find the value of

$$\frac{(a+b)(c+d)-(b+c)(d+a)}{ab+bc-cd-da},$$

when $a=3$, $b=4$, $c=5$ and $d=-4$.

2. Find the value of $(a-x)\,(a-\frac{1}{2}y)\,(a-\frac{1}{4}z)$.

3. If x^2+7x+c is exactly divisible by $x+4$, what is the value of c?

4. Divide $x^2-y^2-z^2+2yz$ by $x+y-z$, and find the coefficient of x in the quotient obtained by dividing $8x^4+xy^3-y^4$ by $x-\frac{1}{2}y$.

5. Shew that

$$(a^2+b^2)\,(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2.$$

6. Shew that

$$(b-c)^2+(a-b)\,(a-c)=(c-a)^2+(b-c)\,(b-a)=(a-b)^2+(c-a)\,(c-b).$$

CHAPTER VI.

SIMPLE EQUATIONS.

75. A STATEMENT of the equality of two algebraical expressions is called an *equation.*

When the statement of equality is true for *all values* of the letters involved, the equation is sometimes called an *identical equation.* An identical equation is however generally called an *identity*, and the name *equation* is reserved for those cases in which the equality is only true for certain particular values of the letters involved.

Thus $a+a=2a$, and $(a+b)^2=a^2+2ab+b^2$, which are true for *all values* of a and b, are *identities;* and $5a+2=12$, which is only true when a is 2, is an *equation.*

NOTE.—For the sake of distinction, a quantity which is supposed to be known, but which is not expressed by any particular arithmetical number, is represented by one of the first letters of the alphabet a, b, c, &c., and a quantity which is unknown, and which is to be found, is represented by one of the last letters of the alphabet x, y or z.

76. To *solve* an equation is to find the value, or values, of the unknown quantity for which the equation is true; and these values of the unknown quantity are said to *satisfy* the equation, and are called the *roots* of the equation.

77. An equation which contains only one unknown quantity, x suppose, is said to be of the *first degree* when x occurs only in the first power; it is said to be of the *second degree* when x^2 is the highest power of x which occurs, and so on.

Equations of the first degree are however generally called *simple equations*, and equations of the second degree are generally called *quadratic equations.*

In the present chapter we shall only consider simple equations which contain one unknown quantity.

78. In the solution of equations frequent use is made of the following *axioms*:—

(i) If we add to equals the same quantity, or equal quantities, the sums will be equal.

(ii) If we take from equals the same quantity, or equal quantities, the differences will be equal.

(iii) If we multiply equals by the same quantity, or by equal quantities, the products will be equal.

(iv) If we divide equals by the same quantity, or by equal quantities, the quotients will be equal.

79. Let

$$a+b=c-d.$$

Add $-b$ to both sides, then the equality still holds good by Axiom (ii);

$$\therefore \quad a+b-b=c-d-b,$$

that is

$$a=c-d-b.$$

Thus the term b has been cancelled from one side of the equation, and it appears on the other side with its sign changed from + to −.

Again, add d to both sides of the last equation; then, by Axiom (i),

$$a+d=c-d-b+d;$$

$$\therefore \quad a+d=c-b.$$

Thus the term d has been cancelled from one side of the equation, and it appears on the other side with its sign changed from $-$ to $+$.

We can proceed in a similar manner in any other case; hence *any term may be moved from one side of an equation to the other, provided its sign is changed.*

When terms are changed from one side of an equation to the other side, they are said to be *transposed.*

80. We may change the signs of *all* the terms of an equation; for, by Axiom (iii), we do not destroy the equality by multiplying both sides, and therefore every term, by -1; and this multiplication will change the sign of every term.

81. We will now solve some equations.

Ex. 1. Solve

$$8x+7=4x+27.$$

Transpose the terms $4x$ and 7, then

$$8x-4x=27-7.$$

Combine like terms, then

$$4x=20.$$

Divide both sides by 4, the coefficient of x, then

$$x=5.$$

Ex. 2. Solve

$$5x-7=7x-15.$$

By transposition, we have

$$5x-7x=-15+7.$$

Combining like terms, we get

$$-2x=-8.$$

Divide both sides by -2, the coefficient of x; then

$$x=4.$$

Ex. 3. Solve $\frac{x}{2}+2=\frac{x}{4}+\frac{5}{2}$.

Multiply every term by 4, to get rid of the fractions; then $$2x+8=x+10.$$

Transposing

$$2x-x=10-8;$$

$$\therefore \quad x=2.$$

Ex. 4. Solve $\frac{x+1}{4}-\frac{1}{3}(x-1)=1$.

Multiply by 12 to get rid of fractions; then

$$\tfrac{12}{4}(x+1)-\tfrac{12}{3}(x-1)=12,$$

that is $$3(x+1)-4(x-1)=12;$$

$$\therefore \quad 3x+3-4x+4=12.$$

Transposing

$$3x-4x=12-3-4;$$

$$\therefore \quad -x=5,$$

or, by changing the signs,

$$x=-5.$$

Ex. 5. Solve $ax+b^2=bx+a^2$.

By transposition,

$$ax-bx=a^2-b^2,$$

that is $$(a-b)x=a^2-b^2.$$

Divide by $a-b$, the coefficient of x; then

$$x=(a^2-b^2)\div(a-b)=a+b.$$

The student should test his results by seeing that the values obtained really *satisfy* the given equations. For example, if we put 5 for x in Ex. 1, we have

$$8\times 5+7=4\times 5+27,$$

that is $$40+7=20+27,$$

which is clearly true. Again, if we put $a+b$ for x in Ex. 5, we have $$a(a+b)+b^2=b(a+b)+a^2,$$

that is $$a^2+ab+b^2=ab+b^2+a^2,$$

which is an identity.

82. The different steps in the process of solving a simple equation are as follows. First, clear the equation of fractions, and perform all the algebraical operations which are indicated. Then transpose all the terms into which the unknown quantity enters to one side of the equation, and all the other terms to the other side. Next combine all the terms which contain the unknown quantity into one term, and divide by the coefficient of the unknown quantity: this gives the required root.

Each fresh form of the equation under discussion should be put in a separate line; and the way in which each new equation is derived from the preceding should always be indicated.

The following are additional examples of simple equations.

Ex. 1. Solve $(x-1)(x-2)+5=(x+1)^2$.

Removing the brackets, we have

$$x^2-3x+2+5=x^2+2x+1.$$

Transposing, we have

$$x^2 - 3x - x^2 - 2x = 1 - 2 - 5;$$

$$\therefore \qquad -5x = -6.$$

Divide by -5; then

$$x = \tfrac{6}{5}.$$

Ex. 2. Solve $3(x-1) - \{3x - (2-x)\} = 5.$

Removing the brackets, we have

$$3x - 3 - 3x + 2 - x = 5;$$

$$\therefore \qquad 3x - 3x - x = 5 + 3 - 2;$$

$$\therefore \qquad -x = 6,$$

or $$x = -6.$$

Ex. 3. Solve $3x^2 - 1 = (3x+2)(x-5).$

Removing the brackets, we have

$$3x^2 - 1 = 3x^2 - 13x - 10.$$

Transposing, we have

$$3x^2 - 3x^2 + 13x = -10 + 1.$$

Hence $$13x = -9,$$

$$\therefore \qquad x = -\tfrac{9}{13}.$$

Ex. 4. Solve $a(x-a) + b(x-b) = 2ab.$

Removing the brackets, we have

$$ax - a^2 + bx - b^2 = 2ab.$$

Transposing, we have

$$ax + bx = a^2 + b^2 + 2ab,$$

that is $$x(a+b) = (a+b)^2;$$

$$\therefore \qquad x = (a+b)^2 \div (a+b) = a + b.$$

EXAMPLES XIII.

Solve the following equations:—

1. $3x+4=x+10.$ 2. $x+7=4x+4.$

3. $5x-12=6x-8.$ 4. $7x+19=5x+7.$

5. $3(x-2)=2(x-3).$ 6. $5(x+2)=3(x+3)+1.$

7. $x-(4-2x)=7(x-1).$ 8. $5(4-3x)=7(3-4x).$

9. $2(x-3)=5(x+1)+2x-1.$ 10. $4(1-x)+3(2+x)=13.$

11. $2(x-2)+3(x-3)+4(x-4)-20=0.$

12. $2(x-1)-3(x-2)+4(x-3)+2=0.$

13. $5x+6(x+1)-7(x+2)-8(x+3)=0.$

14. $x+\frac{2}{3}x=10.$

15. $\frac{x}{5}-\frac{x}{4}=1.$

16. $\frac{x-1}{2}+\frac{x-2}{3}=3.$

17. $\frac{1}{4}(x+1)-\frac{2}{3}(x-1)=3.$

18. $\frac{1}{2}(2-x)-\frac{1}{3}(5x+21)=x+3.$

19. $\frac{1}{2}(x-2)+\frac{1}{3}(x-3)+\frac{1}{4}(x-4)=10.$

20. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+4}{4}+8=0.$

21. $\frac{x-5}{2}-\frac{x-4}{3}=\frac{x-3}{2}-(x-2).$

22. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+3}{4}=16.$

23. $2x-[3-\{4x+(x-1)\}-5]=8.$

24. $1-2\{x-3(1+x)\}=0.$

25. $(x+1)(x+2)=(x-2)(x-4).$

26. $(x-1)(x-2)=(x-3)(x-4).$

27. $2x^2=(x+1)^2+(x+3)^2.$

28. $3x^2=(x+1)^2+(x+2)^2+(x+3)^2.$

29. $(x-2)(x-5)+(x-3)(x-4)=2(x-4)(x-5).$

30. $(x-1)^2+4(x-3)^2=5(x+5)^2.$

31. $5(x+1)^2+7(x+3)^2=12(x+2)^2.$

32. $(x-1)(x-4)=2x+(x-2)(x-3).$

33. $(x-1)^3+(x-2)^3+(x-3)^3=3(x-1)(x-2)(x-3).$

34. $\frac{x+\frac{1}{2}}{2}-\frac{2x-\frac{1}{2}}{5}+1\tfrac{1}{4}=0.$

35. $\frac{3x+5}{8}-\frac{21+x}{2}=5x-15.$

36. $a(x-a)=b(x-b).$

37. $2(x-a)+3(x-2a)=2a.$

38. $\frac{1}{2}(x+a+b)+\frac{1}{3}(x+a-b)=b.$

39. $(a+b)x+(a-b)x=a^2.$

40. $(a+b)x+(b-a)x=b^2.$

41. $\frac{1}{2}(a+x)+\frac{1}{3}(2a+x)+\frac{1}{4}(3a+x)=3a.$

42. $\frac{xa}{b}+\frac{xb}{a}=a^2+b^2.$

43. $(a+bx)(b+ax)=ab(x^2-1).$

44. $(a^2+x)(b^2+x)=(ab+x)^2.$

45. $a(x+a)+b(b-x)=2ab.$

46. $(x+a+b)^2+(x+a-b)^2=2x^2.$

47. $(x-a)(x-b)+(a+b)^2=(x+a)(x+b).$

48. $(x+a+b+c)(x+a-b-c)=(x-a-b+c)(x-a+b-c).$

49. $ax(x+a)+bx(x+b)=(a+b)(x+a)(x+b).$

50. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c).$

CHAPTER VII.

PROBLEMS.

83. We shall in the present chapter consider a class of questions called *problems*. In a problem the magnitudes of certain quantities, some of which are known and others unknown, are connected by given relations; and the values of the unknown quantities have to be found by means of these relations.

In order to solve a problem the relations between the known and unknown quantities must be expressed by means of algebraical symbols: we thus obtain equations, the roots of which are the required values of the unknown quantities.

84. In the present chapter we shall only consider problems in which there is *one* unknown quantity, and in which the relation between the known and the unknown quantities is expressed algebraically by means of a *simple* equation.

Of such problems the following are examples:

Ex. 1. What number is that whose double exceeds its half by 27?

Let x represent the number. Then the double of the number is $2x$, and its half is $\frac{x}{2}$. By the question $2x$ exceeds

$\frac{x}{2}$ by 27. Hence we have the equation

$$2x = \frac{x}{2} + 27.$$

Transposing $$2x - \frac{x}{2} = 27,$$

that is $$\tfrac{3}{2}x = 27;$$

$$\therefore \quad x = 27 \div \tfrac{3}{2} = 18.$$

Thus 18 is the number required.

Ex. 2. *A* has £4 and *B* has 15 shillings. How much must *A* give to *B* in order that he may have just four times as much as *B*?

Let x be the *number of shillings* that *A* gives to *B*.

Then *A* will have $80 - x$ shillings, and *B* will have $15 + x$ shillings. But *A* now has four times as much as *B*. Hence we have the equation

$$80 - x = 4(15 + x).$$

That is $$80 - x = 60 + 4x.$$

Transposing $$-x - 4x = 60 - 80,$$

that is, $$-5x = -20.$$

Divide by -5, then $$x = 4.$$

Thus *A* must give 4 shillings to *B*.

Note. It should be remembered that x must always stand for a *number*. It is also to be noticed that in any problem all concrete quantities of the same kind must be expressed in terms of the same unit; for example, in the above all sums of money were expressed as shillings.

Ex. 3. A line 20 inches long is divided into two parts, one of which is double the other. How long are the parts?

Let x be the *number of inches* in the smaller part. Then, since the two parts together make up 20 inches, the number of inches in the larger part is $20 - x$. By the question the larger part is double the smaller; hence we have the equation

$$20 - x = 2x;$$

$\therefore$ $$20 = 3x,$$

or $$3x = 20.$$

Hence $$x = \tfrac{20}{3} = 6\tfrac{2}{3}.$$

Thus one part is $6\frac{2}{3}$ inches, and the other is $13\frac{1}{3}$ inches.

Ex. 4. A man has 12 coins some of which are half-crowns and the rest shillings, and the coins are worth 18 shillings altogether. How many are there of each kind?

Let x be the number of half-crowns; then $12 - x$ will be the number of shillings. The half-crowns are worth $\frac{5}{2}x$ shillings, and the shillings are worth $12 - x$ shillings. Hence since the coins are worth 18 shillings altogether, we have the equation $$\tfrac{5}{2}x + (12 - x) = 18.$$

Therefore $$\tfrac{5}{2}x - x = 18 - 12,$$

that is $$\tfrac{3}{2}x = 6;$$

$\therefore$ $$x = 6 \div \tfrac{3}{2} = 4.$$

Hence there are 4 half-crowns, and 8 shillings.

Ex. 5. A father is six times as old as his son, and in four years he will be four times as old. How old is each?

Let the son be x years old. Then the father must be $6x$ years old. After four years the son will be $x + 4$, and the father will be $6x+4$ years old. Hence by the question

$$6x + 4 = 4(x + 4),$$

that is
$$6x + 4 = 4x + 16.$$

Hence
$$6x - 4x = 16 - 4,$$

that is
$$2x = 12;$$

$$\therefore \quad x = 6.$$

Hence the son is 6 years old, and the father is 36 years old.

Ex. 6. *A* can do a piece of work in 12 hours, which *B* can do in 4 hours. *A* begins the work, but after a time *B* takes his place, and the whole work is finished in 6 hours from the beginning. How long did *A* work?

Let $x =$ the number of hours that *A* worked.

Then $6 - x =$ the number of hours that *B* worked.

Since *A* can do the whole work in 12 hours, the part done by *A* in 1 hour is $\frac{1}{12}$.

Therefore the part done by *A* altogether is $\dfrac{x}{12}$.

Since *B* can do the whole work in 4 hours, the part done by *B* in 1 hour is $\frac{1}{4}$.

Therefore the part done by *B* altogether is $\frac{1}{4}(6 - x)$.

But *A* and *B* together do the whole of the work. Hence we have the equation

$$\frac{x}{12} + \frac{1}{4}(6 - x) = 1.$$

Multiply by 12, then

$$x + 3(6 - x) = 12,$$

that is
$$x + 18 - 3x = 12.$$

Transposing
$$-2x = -6;$$

$$\therefore \quad x = 3.$$

Hence *A* worked for 3 hours.

Ex. 7. Find the time between 3 and 4 at which the hands of a clock are together.

Suppose that the hands are together at x minutes after 3 o'clock.

At 3 o'clock the hour-hand is 15 minute-spaces in front of the minute-hand, and after x minutes they are together. Hence while the minute-hand moves through x minute-spaces the hour-hand will move through $x-15$ such spaces. But in an hour, and therefore in *any* time, the minute-hand passes over twelve times as many minute-spaces as the hour-hand.

Hence
$$x = 12(x-15),$$
that is
$$x = 12x - 180;$$
$\therefore$
$$11x = 180,$$
or
$$x = \frac{180}{11} = 16\frac{4}{11}.$$

Thus the time required is $16\frac{4}{11}$ minutes past 3 o'clock.

EXAMPLES. XIV.

1. Find two numbers whose sum is 200, and whose difference is 2.

2. Find two numbers whose sum is 56, and whose difference is 20.

3. Divide 25 into two parts whose difference is 5.

4. Divide 100 into two parts whose difference is 45.

5. What number is that to which if you add 40 the sum will be three times the original number?

6. What number is that from which if you take away 14 the remainder is one-third of the original number?

7. The difference of two numbers is 15, and one number is four times the other. Find the numbers.

8. Find two numbers whose difference is 10, and one of which is three times the other.

9. The sum of two numbers is 38, and one of them exceeds twice the other by 2. What are the numbers?

10. Find two numbers the sum of which is 31, and which are such that one of them is less by 2 than half the other.

11. Find a number whose fourth part exceeds its fifth part by 2.

12. Find a number whose third part exceeds its seventh part by 80.

13. Find a number which when multiplied by 4 is as much above 35 as it was originally below it.

14. Find a number which when multiplied by 8 exceeds 27, as much as 27 exceeds the original number.

15. Four times the difference between the fourth and fifth parts of a certain number exceeds by 4 the difference between the third and seventh parts. What is the number?

16. Fifty times the difference between the seventh and eighth parts of a certain number exceeds half the number by 44. What is the number?

17. Divide 100 into two parts such that three times one of the parts plus five times the other is 410.

18. Divide 100 into two parts such that twice one part is equal to three times the other.

19. The difference of two numbers is 20, and one half of one of the numbers is equal to one-fifth of the other. Find them.

20. The sum of two numbers is 36, and their difference is half the greater. Find them.

21. A has £100, and B has £20; how much must A give B in order that B may have half as much as A?

22. A and B play for a stake of *5s.* If A loses he will have as much as B, but if A wins he will have three times as much as B. How much has each?

23. A and B have £50 between them. A wins from B as much as he had originally, and he then had three times as much as B. What had A at first?

24. *A*, *B* and *C* have a certain sum between them. *A* has one half of the whole, *B* has one-third of the whole, and *C* has £50. How much have *A* and *B*?

25. *A* and *B* together have £75, and *A* has £5 more than *B*: how much has each?

26. *A* has £5 more than *B*, *B* has £20 more than *C*, and they have £360 between them. How much has each?

27. *A* has £15 more than *B*, *B* has £5 less than *C*, and they have £65 between them. How much has each?

28. *A* has £5 less than *B*, *C* has as much as *A* and *B* together, and they have £100 between them. How much has each?

29. Divide £30 among 10 men, 20 women and 40 children, giving to each man 15 shillings more than to each child, and to each woman as much as to two children.

30. Divide 15 shillings among 3 men, 5 women and 20 children, giving to each man one shilling more than to each woman, and to each child half as much as to each woman.

31. A man of 40 has a son 10 years old; in how many years will the father be three times as old as the son?

32. One man is 70 and another is 50 years of age; when was the first twice as old as the second?

33. A father's age is three times that of his son, and in 10 years it will be twice as great: how old are they?

34. In 5 years a father will be just four times as old as his daughter, and in 10 years he will be just three times as old: how old is he now?

35. A sum of money is divided between three persons *A*, *B* and *C* in such a way that *A* and *B* have £60 between them, *A* and *C* have £65, and *B* and *C* have £75. How much has each?

36. Four persons *A*, *B*, *C*, *D* are possessed of certain sums of money, such that *A* and *B* together have £49, *A* and *C* together have £51, *B* and *C* together have £53, and *A* and *D* together have £47. How much has each?

37. What is the price of beef if a reduction of 20 per cent. in the price would enable a purchaser to obtain 6 lbs. more for a sovereign?

38. How much are eggs a score, when a rise of 20 per cent. in the price would make a difference of 80 in the number which could be bought for a sovereign?

39. A man leaves one half of his property to his wife, one-third to a son, and the remainder (which is £2000) to a daughter. How much did he leave altogether?

40. A man left his property to be divided between his three children in such a way that the share of the eldest was to be twice that of the second, and the share of the second twice that of the youngest. It was found that the eldest received £750 more than the youngest. How much did each receive?

41. A purse contains 28 coins which altogether amount to £7. A certain number of the coins are shillings, one-fifth of that number are half-sovereigns, and the rest are sovereigns. Find the number of each.

42. A purse contains 36 coins which altogether amount to £11. A certain number of the coins are sovereigns, there are three times as many half-sovereigns, and the rest are shillings. Find the number of each.

43. Find the time between 5 o'clock and 6 o'clock when the hands of a watch are together.

44. Find at what time between 9 and 10 o'clock the hands of a watch are at right angles to one another.

45. There are two numbers one of which exceeds the other by 3, while its square exceeds the square of the other by 99. Find the numbers.

46. In a mixture of spirits and water half of the whole plus 25 gallons was spirit; and a third of the whole minus 5 gallons was water: how many gallons were there of each?

47. A garrison of 1000 men having provisions for 60 days was reinforced after 10 days, and from that time the provisions only lasted 20 days. Find the number in the reinforcement.

48. A labourer was engaged for 36 days, upon the condition that he should receive 2*s*. 6*d*. for every day he worked, but should pay 1*s*. 6*d*. for every day he was idle. At the end of the time he received 58*s*. How many days did he work?

49. A sum of money is divided among three persons. The first receives £10 more than a third of the whole sum; the second receives £15 more than a half of what remains; and the third receives what is over, which is £70. Find the original sum.

50. A purse of sovereigns is divided amongst three persons, the first receiving half of them and one more, the second half of the remainder and one more, and the third 6. Find the number of sovereigns the purse contained.

51. From a sum of money £20 more than its half was taken away: from the remainder £30 more than its third part: and from what then remained £40 more than its fifth part; after which there was nothing left. What was the sum?

52. At a cricket match the contractor provided dinner for 24 persons, and fixed the price so as to gain 12½ per cent. upon his outlay. Three of the cricketers were absent. The remaining 21 paid the fixed price for their dinner, and the contractor lost 1*s*. What was the charge for dinner?

53. *A* can do a piece of work in 30 days, which *B* can do in 20 days. *A* begins the work, but after a time *B* takes his place, and the whole work is finished in 25 days from the beginning. How long did *A* work?

54. *A* can do a piece of work in 20 days, which *B* can do in 30 days. *A* begins the work, but after a time *B* takes his place and finishes it; and *B* worked for 10 days longer than *A*. How long did *A* work?

55. How many men are there in a regiment which can be drawn up in two hollow squares with the men 3 and 5 deep respectively, if the one square will just fit within the other, and the same number of men be in each square?

CHAPTER VIII.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

85. THE *degree* of an equation which contains the two unknown quantities x and y, is the degree of that term, on either side of the equation, which is of the highest dimensions in x and y. [Art. 64.]

Similarly the degree of an equation which contains the unknown quantities x, y and z is the degree of that term which is of the highest dimensions in x, y and z.

Thus $4x+5y=20$ is an equation of the *first degree;* and $3x^2-2xy=7$ is an equation of the *second degree.*

The equations $ax^2+by^2+c=0$, and $x+y=xy$ are also of the *second degree.*

86. A single equation which contains *two* unknown quantities can be satisfied by an indefinite number of pairs of values of the unknown quantities. Consider, for example, the equation $x-3y=24$. It is clear that the equation is satisfied by the values $y=0$, and $x=24$; or $y=1$, and $x=27$; or $y=2$, and $x=30$. In fact, we may suppose y to be equal to *any* quantity k, provided that x is taken equal to $24+3k$.

If there are two equations containing two unknown quantities, each equation taken by itself can, as we have just seen, be satisfied in an indefinite number of ways; but this is not the case when both equations are to be satisfied by the *same values* of the unknown quantities.

Two or more equations which are to be satisfied by the same values of the unknown quantities contained in them, are called *simultaneous equations*.

In the present chapter we shall shew how to solve simultaneous equations of the first degree.

87. In order to solve any two simultaneous equations containing two unknown quantities, we must deduce from the given equations a third equation, from which one of the unknown quantities is absent; and this will give the value, or values, of that unknown quantity which is retained.

88. We will shew how to solve two simultaneous equations of the first degree by taking as an example the following pair of equations:

$$3x + 5y = 22,$$

and $$7x - 4y = 20.$$

Multiply each member of the first equation by 7, and each member of the second equation by 3: the equations then become

$$21x + 35y = 154,$$

and $$21x - 12y = 60.$$

The coefficients of x are now the same in the two equations; hence if we *subtract* each member of the second equation from the corresponding member of the first, we shall obtain an equation from which x is absent: the equation will be

$$47y = 94;$$

whence $$y = 2.$$

If we put this value of y in the first of the given equations, we have $3x+10=22$; hence $3x=22-10$, or $x=4$.

We may if we please find the value of x before finding the value of y. To do this, multiply the first equation by 4, and the second equation by 5: the equations then become

$$12x+20y=88,$$

and

$$35x-20y=100.$$

The coefficients of y in the two equations are now equal in magnitude but opposite in sign; hence if we *add* each member of the second equation to the corresponding member of the first, we shall obtain an equation from which y is absent: the equation will be

$$47x=188;$$

whence

$$x=4.$$

We can now find the value of y by putting $x=4$ in the first of the given equations.

It will be seen from the above example that in order to solve two simultaneous equations of the first degree we can proceed as follows:—Multiply the given equations by numbers such that in the resulting equations the coefficients of one of the unknown quantities may be equal in magnitude; then by addition or subtraction we shall obtain a *simple* equation which contains only *one* of the unknown quantities, and which can be solved as in the preceding chapter.

The unknown quantity which has been got rid of by the above process is said to have been *eliminated.*

It is generally best to eliminate that unknown quantity which has the smaller coefficients in the two equations.

89. The following are additional examples :—

Ex. 1. Solve

$$3x+2y=13, \quad 7x+3y=27.$$

Multiply the equations by 3 and 2 respectively; we then have

$$9x+6y=39,$$

and

$$14x+6y=54.$$

By subtraction we eliminate y, and have

$$-5x=-15;$$

$$\therefore \quad x=3.$$

Put this value of x in the first of the given equations; then

$$9+2y=13;$$

$$\therefore \quad y=2.$$

Thus $x=3$, $y=2$ are the values required.

Ex. 2. Solve

$$2x-3y+14=0, \quad -4x+5y=26.$$

We have

$$2x-3y=-14,$$

and

$$-4x+5y=26.$$

Multiply the first equation by 2; then

$$4x-6y=-28.$$

From the last two equations we have by addition

$$-y=-2,$$

or

$$y=2.$$

We then have from the first equation,

$$2x-6=-14,$$

that is

$$2x=-14+6=-8;$$

$$\therefore \quad x=-4.$$

Hence the values required are

$$x=-4, \quad y=2.$$

Ex. 3. Solve

$$(x-1)(y-2)-(x-2)(y-1)=-2,$$

$$(x+2)(y+2)-(x-2)(y-2)=32.$$

On removing the brackets, the first equation becomes

$$xy-y-2x+2-xy+2y+x-2=-2,$$

or $$-x+y=-2 \quad \text{.....................(i).}$$

The second equation becomes

$$xy+2x+2y+4-xy+2x+2y-4=32,$$

or $$4x+4y=32 \quad \text{.....................(ii).}$$

Multiplying (i) by 4, we have

$$-4x+4y=-8 \quad \text{.................(iii).}$$

From (ii) and (iii) we have by addition

$$8y=24.$$

or $$y=3.$$

From (ii) and (iii) we have by subtraction

$$8x=40,$$

or $$x=5.$$

Hence $x=5$, $y=3$ are the required values.

Ex. 4. Solve

$$\frac{3}{x}+\frac{4}{y}=8, \quad \frac{5}{x}+\frac{6}{y}=13.$$

Multiply the equations by 5 and 3 respectively; then we have

$$\frac{15}{x}+\frac{20}{y}=40,$$

and $$\frac{15}{x}+\frac{18}{y}=39.$$

Hence we have by subtraction

$$\frac{2}{y}=1;$$

$$\therefore \quad y=2.$$

We then have from the first equation

$$\frac{3}{x} + \frac{4}{2} = 8,$$

that is $$\frac{3}{x} = 8 - 2 = 6;$$

$$\therefore \quad 3 = 6x,$$

or $$x = \tfrac{1}{2}.$$

Thus $x = \frac{1}{2}$, $y = 2$ are the values required.

Ex. 5. Solve

$$ax + by = 2ab, \qquad bx - ay = b^2 - a^2.$$

Multiply the first equation by a, and the second equation by b; we then have

$$a^2x + aby = 2a^2b,$$

and $$b^2x - aby = b^3 - a^2b.$$

Hence by addition

$$a^2x + b^2x = a^2b + b^3,$$

that is $$(a^2 + b^2)x = b(a^2 + b^2);$$

$$\therefore \quad x = b.$$

Put this value for x in the first of the given equations; then

$$ab + by = 2ab;$$

$$\therefore \quad by = ab;$$

$$\therefore \quad y = a.$$

Thus $x = b$, $y = a$ are the required values.

90. Instead of proceeding as in Art. 88, we may solve two simultaneous equations in the following manner. We will take as an example the equations $3x - 5y = 2$, and $5x - 2y = 16$.

From the first equation we have $3x = 5y + 2$, and therefore $x = \frac{1}{3}(5y + 2)$. Now substitute this value of x in the second equation, and we get

$$\tfrac{5}{3}(5y + 2) - 2y = 16,$$

which is a simple equation containing only *one* of the unknown quantities. The solution of the last equation gives

$$y = 2.$$

Then, since $y = 2$, we have

$$x = \tfrac{1}{3}(5y + 2) = \tfrac{1}{3}(10 + 2) = 4.$$

We may also proceed as follows. From the first of the given equations we have $3x = 5y + 2$, or $x = \tfrac{1}{3}(5y + 2)$.

Also from the second equation we have $5x = 2y + 16$, or $x = \tfrac{1}{5}(2y + 16)$.

Hence equating the two values of x, we get

$$\tfrac{1}{3}(5y + 2) = \tfrac{1}{5}(2y + 16).$$

From the last equation we find that $y = 2$, and then as before we have $x = 4$.

91. To solve three simultaneous equations of the first degree containing the three unknown quantities x, y and z.

Multiply the first and second of the given equations by such numbers that the coefficients of z in the resulting equations are equal in magnitude: then by addition or subtraction we eliminate z.

Now take the first and third, or the second and third of the given equations, and eliminate z in a similar manner. We then have *two* simultaneous equations to find the values of x and y.

Example, to solve

$$2x + 4y + z = 7 \quad \ldots\ldots\ldots\ldots \text{(i)},$$
$$3x + 2y + 2z = 8 \quad \ldots\ldots\ldots\ldots \text{(ii)},$$
$$5x - 4y + 4z = 9 \quad \ldots\ldots\ldots\ldots \text{(iii)}.$$

Multiply (i) by 2, and we have

$$4x + 8y + 2z = 14 \ldots\ldots\ldots\ldots \text{(iv)}.$$

From (iv) and (ii) we have by subtraction

$$x + 6y = 6 \ldots\ldots\ldots\ldots\ldots\ldots(\text{v}).$$

Now multiply (i) by 4, and we have

$$8x + 16y + 4z = 28 \ldots\ldots\ldots\ldots\ldots(\text{vi}).$$

From (vi) and (iii) we have by subtraction

$$3x + 20y = 19 \ldots\ldots\ldots\ldots\ldots(\text{vii}).$$

We have thus found the two equations (v) and (vii) which contain only x and y.

Now multiply (v) by 3, and we get

$$3x + 18y = 18 \ldots\ldots\ldots\ldots\ldots(\text{viii}).$$

From (vii) and (viii) we have by subtraction

$$2y = 1, \text{ or } y = \tfrac{1}{2}.$$

Then, since $y = \frac{1}{2}$, we have from (v)

$$x = 6 - 6y = 6 - 3 = 3.$$

And, since $y = \frac{1}{2}$, and $x = 3$, we have from (i)

$$z = 7 - 2x - 4y = 7 - 6 - 2 = -1.$$

Hence $x = 3$, $y = \frac{1}{2}$ and $z = -1$ are the values required.

EXAMPLES. XV.

Solve the following equations:—

1. $7x + 4y = 1,$
   $9x + 4y = 3.$

2. $3x + 5y = 19,$
   $5x - 4y = 7.$

3. $x - 11y = 1,$
   $111y - 9x = 99.$

4. $8x - 21y = 5,$
   $6x + 14y = -26.$

5. $\frac{x}{2} + \frac{y}{3} = 1,$
   $\frac{x}{4} - \frac{2y}{3} = 3.$

6. $\frac{x}{3} - \frac{y}{6} = \frac{1}{2},$
   $\frac{x}{5} - \frac{3y}{10} = \frac{1}{2}.$

7. $\frac{x}{3} + 3y + 14 = 0,$
   $\frac{x}{5} + 5y + 4 = 0.$

8. $\frac{x}{5} + 5y = -4,$
   $\frac{y}{5} + 5x = 4.$

9. $$\frac{x+2}{3}+4y=2,\qquad \frac{y+5}{11}-\frac{x+1}{2}=1.$$

10. $$\frac{2x+3y}{5}+\frac{y+6}{7}=2,\qquad \frac{2x-5y}{3}+\frac{x+7}{4}=1.$$

11. $$4x-\tfrac{1}{3}(y-3)=5x-3,\qquad 2y+\tfrac{1}{3}(2x-5)=\frac{21y+37}{6}.$$

12. $$\frac{x}{2}-\tfrac{1}{3}(y-2)-\tfrac{1}{4}(x-3)=0,\qquad x-\tfrac{1}{2}(y-1)-\tfrac{1}{3}(x-2)=0.$$

13. $$\frac{x-2}{3}-\frac{y+2}{4}=0,\qquad \frac{2x-5}{5}-\frac{11-2y}{7}=0.$$

14. $$\frac{x-2}{3}-\frac{y+5}{2}=0,\qquad \frac{2x-7}{3}-\frac{13-y}{16}=0.$$

15. $3x-4y+2=5x-6y-2=7x+2y+4.$

16. $4x-6y-3=7x+2y-4=-2x+3y+24.$

17. $$(x+1)(y+5)=(x+5)(y+1),\qquad xy+x+y=(x+2)(y+2).$$

18. $$xy-(y-1)(x-1)=6(y-1),\qquad x-y=1.$$

19. $$3x+\frac{7y}{2}-2=11y-\frac{2x}{5}=20.$$

20. $$5x+\frac{y}{5}-1=3y+\frac{x}{3}-2=4.$$

21. $$\frac{7+x}{3}=\frac{9+y}{5}=\frac{11+x+y}{7}.$$

22. $$\frac{x+1}{2}=\frac{y+2}{3}=\frac{x+y}{4}.$$

23. $$\frac{9}{x}-\frac{4}{y}=2,\qquad \frac{18}{x}+\frac{8}{y}=10.$$

24. $$\frac{2}{x}+\frac{5}{y}=7,\qquad \frac{3}{x}-\frac{2}{y}=11.$$

25. $$x+\frac{3}{y}=\frac{7}{2},\qquad 3x-\frac{2}{y}=\frac{26}{3}.$$

26. $$2x-\frac{3}{y}=3,\qquad 8x+\frac{15}{y}+6=0.$$

27. $$(a+b)x-(a-b)y=3ab,\qquad (a+b)y-(a-b)x=ab.$$

28. $$bx+ay=2ab,\qquad ax+by=a^2+b^2.$$

29. $$x-y=a-b,\qquad ax-by=2(a^2-b^2).$$

30. $$b^2x-a^2y=0,\qquad bx+ay=a+b.$$

31. $$x+y=a+b,\qquad ax-by+a^2-b^2=0.$$

32. $$(a+b)x+(a-b)y=2ac,\qquad (b+c)x+(b-c)y=2bc.$$

33. $$\begin{aligned} x+2y-3z&=6,\\ 2x+4y-7z&=9,\\ 3x-y-5z&=8. \end{aligned}$$

34. $$\begin{aligned} 2x-y+z&=4,\\ 5x+y+3z&=5,\\ 2x-3y+4z&=20. \end{aligned}$$

35. $$\begin{aligned} x+y+z&=a+b+c,\\ 2x+2y-4z&=2c-a-b,\\ ax+by+cz&=bc+ca+ab. \end{aligned}$$

CHAPTER IX.

PROBLEMS.

92. We shall now give examples of problems which involve more than one unknown quantity, and in which the relations between the known and unknown quantities are expressed algebraically by means of equations of the first degree.

Many of the problems given in Chapter VII. really contain *two* unknown quantities, but the given relations are in those cases of so simple a nature that it is easy to find an equation giving *one* of the unknown quantities in terms of the known quantities, and having found one of the unknown quantities the other is immediately determined.

Ex. 1. Find two numbers such that the greater exceeds twice the less by three, and that twice the greater exceeds the less by 27.

Let x and y be the numbers, of which x is the greater.

Then we have by the question

$$x - 2y = 3,$$

and

$$2x - y = 27.$$

Multiply the first equation by 2; then

$$2x - 4y = 6.$$

Now subtract the members of this last equation from the corresponding members of the second equation, and we have

$$3y = 21,$$

or $$y = 7.$$

Then from the first equation

$$x = 3 + 2y = 3 + 14 = 17.$$

Thus the numbers are 17 and 7.

Ex. 2. A number of two digits is equal to seven times the sum of its digits, and the digit in the ten's place is greater by four than the digit in the unit's place. What is the number?

Let x be the digit in the unit's place, and y be the digit in the ten's place.

Then the number is equal to $10x + y$, and the sum of the digits is $x + y$.

Hence we have

$$10x + y = 7(x + y),$$

that is $$10x + y = 7x + 7y;$$

$\therefore$ $$3x = 6y,$$

or $$x = 2y.$$

We have also

$$x = y + 4.$$

Hence $$2y = y + 4,$$

or $$y = 4,$$

and therefore $$x = 8.$$

Thus the required number is 84.

Ex. 3. Find the fraction which is equal to $\frac{1}{2}$ when its numerator is increased by unity, and is equal to $\frac{1}{3}$ when its denominator is increased by unity.

Let $x =$ the numerator of the fraction, and $y =$ the denominator.

Then we have by the question

$$\frac{x+1}{y} = \tfrac{1}{2},$$

and
$$\frac{x}{y+1} = \tfrac{1}{3}.$$

Multiply the first equation by y; then

$$x + 1 = \frac{y}{2}.$$

Multiply the second equation by $y + 1$; then

$$x = \tfrac{1}{3}(y+1).$$

Subtract the corresponding members of the last two equations; and we have

$$1 = \frac{y}{2} - \tfrac{1}{3}(y+1),$$

from which we find that $y = 8$. Then, since y is 8,

$$x = \frac{y}{2} - 1 = 4 - 1 = 3.$$

Thus the fraction is $\frac{3}{8}$.

Ex. 4. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys. How long would it take one man to do it?

Let $x =$ the number of days in which one man would do the whole;

and let $y =$ the number of days in which one boy would do the whole.

Then a man does $\frac{1}{x}$th of the whole in a day;

and a boy does $\frac{1}{y}$th of the whole in a day.

Now by the question a man and a boy together do $\frac{1}{15}$th of the whole in a day.

Hence we have

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{15}.$$

We have also, since 7 men and 9 boys do half the work in a day

$$\frac{7}{x}+\frac{9}{y}=\frac{1}{2}.$$

Multiply the first equation by 9 and subtract the second; then

$$\frac{9}{x}-\frac{7}{x}=\frac{9}{15}-\frac{1}{2},$$

that is

$$\frac{2}{x}=\frac{1}{10},$$

or

$$\frac{1}{x}=\frac{1}{20};$$

$$\therefore \qquad x=20.$$

Thus one man would do the work in 20 days.

EXAMPLES. XVI.

1. *A* and *B* have £50 between them, but if *A* were to lose half his money, and *B* $\frac{3}{4}$ of his, they would then have only £20. How much has each?

2. A number of two digits has its digits reversed by the addition of 9. Shew that the digits differ by unity.

*3. A man bought 8 cows and 50 sheep for £225. He sold the cows at a profit of 20 per cent., and the sheep at a profit of 10 per cent., and received in all £257. 10*s*. What was the cost of each cow and of each sheep?

4. Twenty-eight tons of goods are to be carried in carts and waggons, and it is found that this will require 15 carts and 12 waggons, or else 24 carts and 8 waggons. How much can each cart and each waggon carry?

5. *A* and *B* can perform a certain task in 30 days, working together. After 12 days however *B* is called off, and *A* finished it by himself 24 days after. How long would each take to do the work alone?

6. If the numerator of a certain fraction be increased by 1 and its denominator diminished by 1, its value will be 1. If the numerator be increased by the denominator and the denominator be diminished by the numerator, its value will be 4. Find the fraction.

7. Find the fraction such that if you quadruple the numerator and add 3 to the denominator the fraction is doubled, but if you add 2 to the numerator and quadruple the denominator the fraction is halved.

8. The first edition of a book had 600 pages, and was divided into two parts. In the second edition one quarter of the second part was omitted and 30 pages added to the first. The change made the two parts of the same length. What were they in the first edition?

9. If *A* were to receive £10 from *B* he would then have twice as much as *B* would have left; but if *B* were to receive £10 from *A*, *B* would have three times as much as *A* would have left. How much has each?

10. A farmer sold 30 bushels of wheat and 50 bushels of barley for £18. 15*s*. He also sold at the same prices 50 bushels of wheat and 30 bushels of barley for £19. 5*s*. What was the price of the wheat per bushel?

11. A rectangle is of the same area as another which is 6 yards longer and 4 yards narrower; it is also of the same area as a third, which is 8 yards longer and 5 yards narrower. What is its area?

12. *A* and *B* can together do a piece of work in 15 days. After working together for 6 days, *A* went away, and *B* finished it by himself 24 days after. In what time would *A* alone do the whole?

13. An income of £120 a year is derived from a sum of money invested partly in $3\frac{1}{2}$ per cent. stock and partly in 4 per cent. stock. If the stock be sold out when the $3\frac{1}{2}$ per cents. are at 108 and the 4 per cents at 120, the capital realised is £3672. How much stock of each kind was there?

14. A number of two digits is equal to seven times the sum of its digits: shew that one digit must be twice the other.

15. Find all the numbers of two digits, each of which is equal to four times the sum of the digits.

16. £1000 is divided between *A*, *B*, *C* and *D*. *B* gets half as much as *A*; the excess of *C*'s share over *D*'s share is equal to one-third of *A*'s share, and if *B*'s share were increased by £100 he would have as much as *C* and *D* have between them. Find how much each gets.

17. A number has two digits of which the second is double of the first; and, if the digits be reversed, the new number exceeds the original number by 36; find the number.

18. A certain number consists of two digits, and another number is found from it by reading it backwards. If the sum of the two numbers is 99 and the difference is 45, find the digits.

19. In a certain proper fraction the difference between the numerator and the denominator is 12, and if each be increased by 5 the fraction becomes equal to $\frac{3}{4}$. Find it.

20. The wages of 10 men and 8 boys for a day amount to £1. 17*s.*; and four men receive one shilling more than six boys. How much does each boy receive?

21. A farmer has two farms for each of which he pays a rent of 30 shillings an acre, and his total rent is £675. If the rent of one farm were reduced by 5 shillings an acre, and that of the other by 10 shillings an acre, his rent would be £500. What is the acreage of each of the farms?

22. A man has one pound's worth of silver in half-crowns, shillings and sixpences; and he has in all 20 coins. If he changed the sixpences for pennies, and the shillings for sixpences, he would have 73 coins. How many coins of each kind has he?

23. The price of a passenger's ticket on a French railway is proportional to the distance he travels; he is allowed 25 kilogrammes of luggage free, but on every kilogramme beyond this amount he is charged a sum proportional to the distance he goes. If a journey of 200 miles with 50 kilos. of luggage cost 25 francs, and a journey of 150 miles with 35 kilos. cost $16\frac{1}{2}$ francs, what will a journey of 100 miles with 100 kilos. cost?

24. *A* has twice as many pennies as shillings; *B*, who has 8*d*. more than *A*, has twice as many shillings as pennies; together they have one more penny than they have shillings. How much has each?

25. Of the candidates in a certain examination one quarter fail. The number of marks required for passing is less by 2 than the average marks obtained by all the candidates, less by 11 than the average marks of those who pass, and equal to double the average marks of those who fail. How many marks are required for passing?

MISCELLANEOUS EXAMPLES. II.

A.

1. Find the value of

$$\frac{ab+2bc-3cd}{b+c+d}+\frac{a^3+b^3-c^3}{a^2+b^2-c^2}, \text{ when } a=1,\ b=2,\ c=-3 \text{ and } d=0.$$

2. Subtract $2a-3(a-\overline{b-a})$ from $2b-3(b-\overline{a-b})$.

3. Shew that

$$(x+3)(y+3)-3(x+1)(y+1)+3(x-1)(y-1)-(x-3)(y-3)=0,$$

and that

$$(x+2)(y+2)-4(x+1)(y+1)+6xy-4(x-1)(y-1)+(x-2)(y-2)=0.$$

4. Divide $x^5-5x^4+7x^3-x^2-4x+2$ by x^3-3x^2+3x-1.

5. Solve

(i) $\frac{x}{2}+\frac{x+1}{3}+\frac{x+2}{4}+\frac{x+3}{5}=4$.

(ii) $\frac{x}{3}+\frac{y}{5}=5,\ 2x+\frac{y}{3}-17=0$.

(iii) $\left.\begin{aligned}(a+b)x+(a-b)y=a^2-b^2\\(a-b)x+(a+b)y=a^2+b^2\end{aligned}\right\}$.

6. A number of marbles was divided among three boys so that the first boy had 10 less than half of the whole, the second had 10 more than a quarter of what was left, and the third had 20. How many marbles were there?

B.

1. Find the numerical value of

$$(a-b)^2+(b-c)^2+(c-a)^2+(b-c)(c-a)(a-b),$$

when $a=1$, $b=2$ and $c=\frac{1}{2}$.

2. From the sum of $(2a-b)^2$ and $(a-2b)^2$ take the square of $2(a-b)$.

3. Shew that

$$n^3=n(n-1)(n-2)+3n(n-1)+n,$$

and $$n^4=n(n-1)(n-2)(n-3)+6n(n-1)(n-2)+7n(n-1)+n.$$

4. Find the algebraical expression which when divided by x^2-2x+1 gives a quotient x^2+2x+1 and a remainder $x+1$.

5. Solve

(i) $3(x+3)^2+5(x+5)^2=8(x+8)^2.$

(ii) $$\left.\begin{aligned}\tfrac{1}{2}(x+y)-\tfrac{1}{3}(x-y)&=8\\ \tfrac{1}{3}(x+y)-\tfrac{1}{4}(x-y)&=11\end{aligned}\right\}.$$

(iii) $\dfrac{3}{x}+\dfrac{4}{y}=1,\quad \dfrac{7}{x}-\dfrac{8}{y}=\dfrac{1}{6}.$

6. A man paid a bill of £100 with sovereigns and crowns, using in all 130 coins. How many coins of each sort did he use?

C.

1. Simplify $a-[a+b-\{a+b+c-\overline{a+b+c+d}\}]$.

2. Find the sum, the difference, and the product of $a+b$ and $\dfrac{a}{2}-\dfrac{b}{2}$.

3. Shew that

$$ab-3(a-1)(b-1)+3(a-2)(b-2)-(a-3)(b-3)=0,$$

and that

$$ab-4(a-1)(b-1)+6(a-2)(b-2)-4(a-3)(b-3)+(a-4)(b-4)=0.$$

4. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$, and from the result write down, without division, the quotient when $x^3+y^3-1+3xy$ is divided by $x+y-1$.

5. Solve

(i) $x=\frac{1}{3}(5y+2)$, $y=\frac{1}{2}(x-1)$.

(ii) $(b+c)x+(b-c)y=2ab$,
$(c+a)x+(c-a)y=2ac$.

6. A and B each shoot 30 arrows at a target. B makes twice as many hits as A, and A makes three times as many misses as B. Find the number of hits and misses of each.

D.

1. Find the value of

$$abc\,(ab+bc+cd+da)\div(a+b)\,(a+c)\,(a+d),$$

when $a=1$, $b=3$, $c=-5$ and $d=0$.

2. Simplify $(a+b)^2-[2a^2-\{(a-b)\,(a+2b)-b\,(a-b)\}]$.

3. Find the continued product of $x+a$, $x+b$ and $x+c$; and from the result *write down* the continued product of $a-x$, $a-y$ and $a-z$.

4. Divide $75a^2b^3c^5-15a^3b^4c^3$ by $5a^2bc^2$, and

$$a^2-2b^2-6c^2+ab-ac+7bc \text{ by } a-b+2c.$$

5. Solve

(i) $\frac{1}{2}(x-1)-\frac{1}{3}(x-2)+\frac{1}{4}(x-3)=0$.

(ii) $\dfrac{x-3}{2}=\dfrac{y-1}{3}=\dfrac{x+y}{9}$.

(iii) $\dfrac{3}{x}+\dfrac{5}{y}=22$, $\dfrac{7}{x}-\dfrac{4}{y}=20$.

6. A square grass plot would contain 69 square feet less if each side were one foot shorter. How many square feet does it contain?

E.

1. Simplify

$$b^2+\{b\,(a-c)+ac\}-a\,\{b-\overline{a-c}\}+b\,(c-\overline{b-a})-ab.$$

2. From what must the sum of $3a^2+2ab-ac$, $3b^2+2bc-ba$ and $3c^2+2ca-cb$ be taken in order that the remainder may be $a^2+b^2+c^2$?

3. Shew that $(x+y)^2+(y+z)^2+(z+x)^2$

$$+2\,(x+y)\,(x+z)+2\,(y+z)\,(y+x)+2\,(z+x)\,(z+y)$$
$$=4\,(x+y+z)^2.$$

4. Divide $\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{3}x+\frac{4}{3}$ by $\frac{1}{3}x+2$, and

$acx^3+(ad-bc)x^2-(ac+bd)x+bc$ by $ax-b$.

5. Solve

(i) $(x-1)(x+2)=x^2+3$.

(ii) $\left.\begin{array}{r}2x+\cdot 4y=1\cdot 2\\ 5x+\cdot 2y=1\cdot 8\end{array}\right\}$.

(iii) $\left.\begin{array}{r}(a-b)x=(a+b)y\\ x+y=2a\end{array}\right\}$.

6. Find two numbers such that twice their difference is greater by unity than the smaller number, and is less by two than the larger number.

F.

1. Find the value of $3(x+y+z)^3-(x^3+y^3+z^3)$, when $x=3$, $y=-5$ and $z=7$.

2. Find the continued product of $a-x$, $a+x$, a^2+x^2, and a^4+x^4.

3. Shew that $(1-a)(1-b)+a(1-b)+b=1$,
and that $(1-a)(1-b)(1-c)+a(1-b)(1-c)+b(1-c)+c=1$.

4. Divide $4y^4-9y^2+6y-1$ by $2y^2+3y-1$.

Divide also $a^2+2b^2-3c^2+bc+2ac+3ab$ by $a+b-c$.

5. Solve

(i) $(a+x)(b+x)=(c+x)(d+x)$.

(ii) $3x-7y=4$, $7x-9y=2$.

6. If 12 lbs. of tea and 3 lbs. of coffee cost £1. 15*s*., and 12 lbs. of coffee and 3 lbs. of tea cost £1. 7*s*. 6*d*., what is the price per lb. of tea and of coffee?

CHAPTER X.

FACTORS.

93. IN the chapter on Division we supposed the product of two factors given, and also one of the factors, and we shewed how to find the other factor. In the present chapter we shall consider certain simple cases in which the factors of an expression can be found when none of the factors are given. To do this we shall compare the expression with some known result of multiplication.

An algebraical expression which does not contain any *letter* in the denominator of any term is said to be an *integral expression.* Thus, $\frac{1}{2}ax + \frac{1}{4}b$ is an integral expression; but $\frac{x^2}{a} + b$ and $\frac{x+1}{x+2}$ are not integral expressions.

An expression is said to be integral with respect to any particular letter, when that letter does not occur in the denominator of any term of the expression: thus $\frac{x^2}{a} + \frac{x}{a+b}$ is integral *with respect to x.*

An expression is said to be rational when none of its terms contain square or other roots. [Art. 11.]

We shall only consider the factors of rational and integral expressions.

In arithmetic we mean by the factors of a number, its integral divisors only; similarly, by the factors of an algebraical expression, we mean the rational and integral expressions which exactly divide it.

94. The simplest case is when some letter is common to all the terms of the expression whose factors are required. In this case each term, and therefore the whole expression, is divisible by that letter.

Thus $$ab + ac = a(b + c),$$

$$a^2b + ab^2 = ab(a + b),$$

and $$3ax^3y + 6axy^3 = 3axy(x^2 + 2y^2).$$

We can thus find by inspection the *simple factors* of any expression, if there are any; and it is always best to do this before attempting to find the compound factors. In what follows therefore we need only consider expressions which have no simple factors.

95. We have seen [Art. 59] that

$$a^2 + 2ab + b^2 = (a + b)^2,$$

and that

$$a^2 - 2ab + b^2 = (a - b)^2.$$

Hence, when a trinomial expression consists of the sum of the squares of any two quantities *plus* twice the product of the quantities, it is equal to the square of their *sum*. And when a trinomial expression consists of the sum of the squares of any two quantities *minus* twice their product, it is equal to the square of their *difference*.

Hence it is easy to recognise when a trinomial is a complete square; for two of the terms must be squares, and the remaining term must be twice the product of the quantities whose squares are the other terms.

It should be noticed that the two terms which are squares must both be positive; the middle term may however be positive or negative.

Thus $a^2+6ab+9b^2$, that is $a^2+(3b)^2+2a\,(3b)$, consists of the squares of a and $3b$ together with twice the product of a and $3b$, and hence is equal to $(a+3b)^2$.

Also $4a^2-4ab+b^2$, that is $(2a)^2-2\,(2a)\,b+b^2$, is equal to $(2a-b)^2$.

As other examples we have

$$16a^4+8a^2+1=(4a^2+1)^2,$$

$$x^2-4xy^2+4y^4=(x-2y^2)^2,$$

and $$(a+b)^2-2c\,(a+b)+c^2=\{(a+b)-c\}^2=(a+b-c)^2.$$

NOTE. We may consider that $a^2-2ab+b^2$ is equal to $(b-a)^2$ instead of $(a-b)^2$, for by the Law of Signs $(b-a)^2$, that is $\{-(a-b)\}^2$, is equal to $(a-b)^2$. In fact whenever we find that an expression is equal to the product of two factors, we may equally well consider it as the product of the same two factors with all the signs changed.

EXAMPLES. XVII.

Find the factors of each of the following expressions:—

1. $9a^2-6ab+b^2$.
2. $4x^2-12x+9$.
3. $x^2+xy+\frac{y^2}{4}$.
4. $4x^2-xy+\frac{y^2}{16}$.
5. $4a^2x^2+4abxy+b^2y^2$.
6. $3+6a^2x+3a^4x^2$.
7. $4ab-a^2-4b^2$.
8. $8x-4x^2-4$.
9. $4x^2y^2+2xy^3+\frac{y^4}{4}$.
10. $x^3y-4x^2y^2+4xy^3$.
11. $4x^2y^2+4\,(a+b)\,xy+(a+b)^2$.
12. $9\,(a+b)^2-6c\,(a+b)+c^2$.

96. From the *formula*

$$a^2 - b^2 = (a + b)(a - b),$$

we see that the difference of the squares of any two quantities is equal to the product of the sum and the difference of the quantities.

Thus $a^2 - 4b^2$, or $a^2 - (2b)^2$, is equal to $(a + 2b)(a - 2b)$.

Also $9x^4 - 4x^2y^2$, that is $x^2\{(3x)^2 - (2y)^2\}$ is equal to $x^2(3x + 2y)(3x - 2y)$;

and $a^2 - 2b^2$, that is $a^2 - (\surd 2b)^2$, is equal to

$$(a + \surd 2b)(a - \surd 2b)^*.$$

As other examples, we have:

$$\begin{aligned} 8axy - 18a^3x^3y^3 &= 2axy(4 - 9a^2x^2y^2) \\ &= 2axy(2 - 3axy)(2 + 3axy), \end{aligned}$$

$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b),$$

and $\quad 79^2 - 71^2 = (79 + 71)(79 - 71) = 150 \times 8 = 1200.$

We may deal with the squares of compound expressions in precisely the same way as with the squares of simple expressions. Thus

$$\begin{aligned} (a + b)^2 - c^2 &= \{(a + b) + c\}\{(a + b) - c\} \\ &= (a + b + c)(a + b - c), \end{aligned}$$

$$a^4 + a^2b^2 + b^4 = (a^2 + b^2)^2 - (ab)^2 = (a^2 + b^2 + ab)(a^2 + b^2 - ab),$$

and

$$\begin{aligned} &(2a - b + 2c)^2 - (a + 4b + c)^2 \\ &\quad = \{(2a - b + 2c) + (a + 4b + c)\}\{(2a - b + 2c) - (a + 4b + c)\} \\ &\quad = 3(a + b + c)(a - 5b + c). \end{aligned}$$

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* The expressions $a + \surd 2b$ and $a - \surd 2b$ are rational so far as the *letters* are concerned.

EXAMPLES. XVIII.

Find the factors of each of the following expressions:—

1. $a^4 - b^2$.	2. $x^2 - 9y^4$.
3. $a^2 - 2$.	4. $x^2 - 3y^2$.
5. $a^4 - 16b^4$.	6. $81x^4y^4 - a^4$.
7. $x^3y - 4xy^3$.	8. $4x^2y - 9y^5$.
9. $3x^5 - 27a^2x^3$.	10. $20a^3x^3 - 45axy^2$.
11. $(a - 2b + 3c)^2 - (a - c)^2$.	12. $(3x^2 + x - 2)^2 - (x^2 - x - 2)^2$.

97. From the *formulae*

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2),$$

and

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2), \qquad \text{[Art. 72]}$$

we see that the *sum of the cubes* of any two quantities is divisible by the *sum* of the quantities; and that the *difference of the cubes* of any two quantities is divisibl by thee *difference of the quantities.*

Thus $8a^3 + 27b^3$, that is $(2a)^3 + (3b)^3$, is divisible by $2a + 3b$, and the quotient is $(2a)^2 - (2a)(3b) + (3b)^2$, or $4a^2 - 6ab + 9b^2$. Also $27x^3 - 8$, that is $(3x)^3 - 2^3$, is equal to $(3x - 2)\{(3x)^2 + (2 . 3x) + 2^2\} = (3x - 2)(9x^2 + 6x + 4)$.

As other examples,

$$a^3b^3 - \tfrac{1}{8}c^3 = (ab - \tfrac{1}{2}c)(a^2b^2 + \tfrac{1}{2}abc + \tfrac{1}{4}c^2),$$

and

$$\begin{aligned} a^6 - b^6 &= (a^3 + b^3)(a^3 - b^3) \\ &= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2). \end{aligned}$$

The cubes of compound expressions may be dealt with in precisely the same way. Thus

$$(a + b)^3 - c^3 = \{(a + b) - c\}\{(a + b)^2 + (a + b)c + c^2\},$$

and $(x - 2y)^3 - (y - 2x)^3$

$$= (\overline{x - 2y} - \overline{y - 2x})\{(x - 2y)^2 + (x - 2y)(y - 2x) + (y - 2x)^2\}.$$

EXAMPLES. XIX.

Find the factors of each of the following expressions:—

1. $4a^3+32b^3$. 2. $27x^3y^3-\frac{1}{8}z^3$.

3. a^3-8a^6. 4. $8a^6+b^3$.

5. $2xy^3-\frac{1}{4}x^4$. 6. $9a^4b^2-\frac{1}{3}ab^5$.

7. $3a^4b+24ab^4$. 8. $40a^3bc-5b^4c^4$.

9. a^6-64. 10. $x^{12}-a^6b^6$.

11. $(x+2y)^3+(y+2x)^3$. 12. $(x-3y)^3-(y-3x)^3$.

98. By multiplication we have

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

Hence conversely, if an expression of the form x^2+px+q be the product of the two factors $x+a$ and $x+b$, the given expression must be the same as $x^2+(a+b)x+ab$; we must therefore have $p=a+b$, and $q=ab$. Hence *a and b are such that their sum is p, and their product q.*

For example, to find the factors of $x^2+7x+12$. The factors will be $x+a$ and $x+b$, where $ab=12$, and $a+b=7$. Hence we must find two numbers whose product is 12, and whose sum is 7. Pairs of numbers whose product is 12 are 12 and 1, 6 and 2, and 4 and 3; and the sum of the last pair is 7. Hence $x^2+7x+12=(x+4)(x+3)$.

Again, to find the factors of $x^2-7x+10$. We have to find two numbers whose product is 10, and whose sum is -7. Since the product is $+10$, the two numbers are *both* positive or *both* negative; and since the sum is -7, they must both be negative. The pairs of negative numbers whose product is 10 are -10 and -1, and -5 and -2; and the sum of the last pair is -7. Hence $x^2-7x+10=(x-5)(x-2)$.

Again, to find the factors of $x^2+3x-18$. We have to find two numbers whose product is -18, and whose sum is 3.

Since the product is -18, one of the numbers is positive and the other negative. The pairs of numbers whose product is -18 are -18 and 1, -9 and 2, -6 and 3, -3 and 6, -2 and 9, and -1 and 18; and of these pairs the sum of 6 and -3 is 3. Hence $x^2+3x-18=(x+6)(x-3)$.

The above method of finding the factors of an expression of the form x^2+px+q is only applicable in certain cases. For example, if we have to find the factors of x^2+6x+7 we must find two numbers whose product is 7, and whose sum is 6; and no integral numbers will satisfy these two conditions, nor can we guess any other numbers which will satisfy the conditions.

It should be noticed that the factors of $x^2-5xy+4y^2$ can be found in the same way as the factors of x^2-5x+4. For we must find two quantities whose product is $4y^2$, and whose sum is $-5y$: these are $-4y$ and $-y$. Hence

$$x^2-5xy+4y^2=(x-4y)(x-y).$$

EXAMPLES. XX.

Find the factors of each of the following expressions:—

1. x^2-4x+3.
2. $x^2+9x+20$.
3. $x^2-9x+14$.
4. $x^2-23x+132$.
5. $x^2+2x-35$.
6. $x^2-3x-10$.
7. $x^2+5x-14$.
8. $x^2-x-132$.
9. $x^2+4xy+4y^2$.
10. $x^2-5xy+4y^2$.
11. $4x^2-5xy^2+y^4$.
12. $10x^2+3x-1$.
13. $9x+10-x^2$.
14. $2x^2+8-x^4$.
15. x^3-3x^2-18x.
16. $x^3y-x^2y^2-2xy^3$.

99. Since $x^2+2ax+a^2$ is a perfect square, namely $(x+a)^2$, it follows that in order to complete the square of which x^2 and $2ax$ are the first two terms, we must add the square of a; that is we must *add the square of half the coefficient of x.*

For example, x^2+6x is made a perfect square by the addition of 3^2; and the square is x^2+6x+9, which is $(x+3)^2$.

So also, x^2+5x is made a perfect square by the addition of $(\frac{5}{2})^2$; and the square is $x^2+5x+\frac{25}{4}$, which is $(x+\frac{5}{2})^2$.

And x^2-7x is made a perfect square by the addition of $(-\frac{7}{2})^2$, that is of $\frac{49}{4}$; and the square is $x^2-7x+\frac{49}{4}$, which is $(x-\frac{7}{2})^2$.

Whatever a may be, x^2+ax is made a perfect square by the addition of $\left(\frac{a}{2}\right)^2$, for

$$x^2+ax+\left(\frac{a}{2}\right)^2 \text{ is } \left(x+\frac{a}{2}\right)^2.$$

100. We will now find the factors of x^2+ax+b by a method which is applicable in all cases.

The required factors are to be rational and integral *with respect to x.*

We first apply the method to some examples.

Ex. 1. Find the factors of

$$x^2+6x+8.$$

As we have seen in the last Article, x^2+6x is made a perfect square by the addition of 3^2; hence

$$\begin{aligned} x^2+6x+8 &= x^2+6x+9-9+8 \\ &= (x+3)^2-1. \end{aligned}$$

The last form of the expression is *the difference of two squares;* and the factors are therefore $\overline{x+3}+1$ and $\overline{x+3}-1$, that is $x+4$ and $x+2$. Hence the factors of x^2+6x+8 are $x+4$ and $x+2$.

Ex. 2. Find the factors of $x^2-3x-28$.

x^2-3x is made a perfect square by the addition of $(-\frac{3}{2})^2$, that is of $\frac{9}{4}$; hence

$$\begin{aligned} x^2-3x-28 &= x^2-3x+\tfrac{9}{4}-\tfrac{9}{4}-28 \\ &= (x-\tfrac{3}{2})^2-\tfrac{121}{4} \\ &= (x-\tfrac{3}{2})^2-(\tfrac{11}{2})^2. \end{aligned}$$

Now the factors of the last expression are $x-\frac{3}{2}+\frac{11}{2}$ and $x-\frac{3}{2}-\frac{11}{2}$.

Thus the required factors are $x+4$ and $x-7$.

Ex. 3. To find the factors of x^2+6x+7.

x^2+6x is made a perfect square by the addition of 3^2; hence

$$\begin{aligned} x^2+6x+7 &= x^2+6x+9-9+7 \\ &= (x+3)^2-2 \\ &= (x+3)^2-(\surd 2)^2. \end{aligned}$$

Now the factors of the last expression, and therefore of x^2+6x+7, are $x+3+\surd 2$ and $x+3-\surd 2$.

Ex. 4. To find the factors of $3x^2-10x+3$.

$$3x^2-10x+3 = 3\,(x^2-\tfrac{10}{3}x+1).$$

Now $x^2-\frac{10}{3}x$ is made a perfect square by the addition of $(-\frac{5}{3})^2$, that is of $\frac{25}{9}$; hence

$$\begin{aligned} x^2-\tfrac{10}{3}x+1 &= x^2-\tfrac{10}{3}x+\tfrac{25}{9}-\tfrac{25}{9}+1 \\ &= (x-\tfrac{5}{3})^2-\tfrac{16}{9} \\ &= (x-\tfrac{5}{3})^2-(\tfrac{4}{3})^2. \end{aligned}$$

The factors of the last expression are $x-\frac{5}{3}+\frac{4}{3}$ and $x-\frac{5}{3}-\frac{4}{3}$, that is $x-\frac{1}{3}$ and $x-3$. Hence

$$3x^2-10x+3=3\left(x-\tfrac{1}{3}\right)(x-3).$$

From the above examples it will be seen that the method of finding the factors of x^2+ax+b consists in changing the expression into an equivalent one which is *the difference of two squares.*

Since x^2+ax is made a perfect square, namely $\left(x+\frac{a}{2}\right)^2$, by the addition of $\left(\frac{a}{2}\right)^2$, we have

$$\begin{aligned}x^2+ax+b&=x^2+ax+\left(\frac{a}{2}\right)^2-\frac{a^2}{4}+b\\&=\left(x+\frac{a}{2}\right)^2-\left(\frac{a^2}{4}-b\right)\\&=\left(x+\frac{a}{2}\right)^2-\left\{\sqrt{\left(\frac{a^2}{4}-b\right)}\right\}^2.\end{aligned}$$

The factors of the last expression are, by Art. 96,

$$x+\frac{a}{2}+\sqrt{\left(\frac{a^2}{4}-b\right)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and $$x+\frac{a}{2}-\sqrt{\left(\frac{a^2}{4}-b\right)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)};$$

and these are the factors required.

101. Instead of working out every example from the beginning, we might use the general formulae (i) and (ii) found in the last article; and we should only have to substitute for a and b their values in the particular case we are considering.

Thus, to find the factors of $x^2+7x+12$.

Here $a=7$, $b=12$. Hence the factors are

$$x+\tfrac{7}{2}+\sqrt{(\tfrac{49}{4}-12)} \text{ and } x+\tfrac{7}{2}-\sqrt{(\tfrac{49}{4}-12)};$$

that is, $$x+\tfrac{7}{2}+\tfrac{1}{2} \text{ and } x+\tfrac{7}{2}-\tfrac{1}{2},$$

or $$x+4 \text{ and } x+3.$$

Again, to find the factors of x^2+6x+7.

Here $a=6$ and $b=7$.

Hence the factors are

$$x+\tfrac{6}{2}+\sqrt{(\tfrac{36}{4}-7)} \text{ and } x+\tfrac{6}{2}-\sqrt{(\tfrac{36}{4}-7)};$$

that is, $$x+3+\sqrt{2} \text{ and } x+3-\sqrt{2}.$$

102. If in the formulae (i) and (ii) of Art. 100, we put 2 for a and 4 for b, we obtain the factors of the expression x^2+2x+4; these factors are $x+1+\sqrt{(1-4)}$ and $x+1-\sqrt{(1-4)}$, that is $x+1+\sqrt{(-3)}$ and $x+1-\sqrt{(-3)}$.

Now all squares, whether of positive or of negative quantities, are *positive.* Hence it is impossible to find any quantity whose square is -3. Such an expression as $\sqrt{-3}$ is however often used in Algebra; and the meaning to be given to $\sqrt{-3}$ is simply that expressed by $\sqrt{-3}\times\sqrt{-3}=-3$.

The square root of a negative quantity is said to be *imaginary* or *impossible*; and $x+1+\sqrt{-3}$ and $x+1-\sqrt{-3}$ are said to be imaginary factors of x^2+2x+4.

By referring to the formulae (i) and (ii) of Art. 100, it will be seen that the factors of x^2+ax+b are imaginary whenever a^2-4b is negative.

103. The factors of *any* expression of the third degree, or of the fourth degree, can be found; the methods are, however, beyond the range of this book.

Expressions of higher degree than the fourth cannot, except in special cases, be resolved into factors.

The expression $x^3+y^3+z^3-3xyz$ is of frequent occurrence, and its factors should be known. From Art. 73 we have

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy).$$

The expression x^4+x^2+1 of the fourth degree is also of frequent occurrence. Writing it in the form $(x^2+1)^2-x^2$, we see that x^2+1+x and x^2+1-x are factors; and we do not proceed any further, for both x^2+x+1 and x^2-x+1 have imaginary factors of the first degree.

EXAMPLES. XXI.

Find the factors of each of the following expressions:—

1. $16x^4-625y^4$.
2. $81a^4-16b^4$.
3. $1+27a^3$.
4. $27+8x^3$.
5. $a^2-b^2-(a-b)^2$.
6. $(a+b)^2-4(c-d)^2$.
7. $a^2+ab-bc-c^2$.
8. $a^4+a^2b^2-b^2c^2-c^4$.
9. $(a+b)^2-(a+c)^2$.
10. $(x+y)^4-(x-y)^4$.
11. $(a+b+c)^2-4c^2$.
12. $a^2+b^2-c^2-2ab$.
13. $(a^2+ab+b^2)^2-(a^2-b^2)^2$.
14. $(2x+3y)^3+(2y+3x)^3$.
15. $x^2-\frac{3}{4}xy-\frac{1}{4}y^2$.
16. $x^2+\left(a+\dfrac{1}{a}\right)x+1$.
17. $x^2+\frac{13}{6}x+1$.
18. $x^2-\frac{5}{6}x-1$.
19. $6x^2-5xy-6y^2$.
20. $72(x^2-1)-17x$.
21. x^4-5x^2+4.
22. $x^4-13x^2y^2+36y^4$.
23. $mn(x^2+y^2)+xy(m^2+n^2)$.
24. $2x^2+7x-39$.
25. $(a+b)^2-5x(a+b)+6x^2$.
26. $(x+y)^2-8(a+b)(x+y)+15(a+b)^2$.
27. $(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2$.
28. $(x^2+4x)^2-2(x^2+4x)-15$.
29. $a^2-b^2+c^2-d^2-2(ac-bd)$.
30. $4a^2b^2-(a^2+b^2-c^2)^2$.
31. $9a^2+6ab-4c^2+b^2$.
32. $a^4-a^3y+az^3-yz^3$.

CHAPTER XI.

HIGHEST COMMON FACTORS.

104. A *common factor* of two or more algebraical expressions is an expression which will exactly divide each of them.

The *Highest Common Factor* of two or more algebraical expressions is the expression of *highest dimensions* which will exactly divide each of them.

Instead of Highest Common Factor it is usual to write H. C. F.

We proceed to shew how to find the H. C. F. of given expressions.

105. We first consider *simple* expressions, the factors of which can be seen by inspection.

Take, for example, a^3b^2c and $a^2b^3c^4$.

The first expression is clearly divisible by a, or by a^2, or by a^3, but by no higher power of a; and the second expression is divisible by a, or by a^2, but by no higher power. Hence a^2 will divide both expressions, and it is the highest power of a which will divide them both. Also b^2, but no higher power of b, will divide both expressions; and c, but no power of c above the first, will divide both expressions.

Hence the H. C. F. of a^3b^2c and $a^2b^3c^4$ is a^2b^2c.

Again, to find the H. C. F. of $ab^3c^3d^2$ and $a^3c^5d^3$.

The highest power of a which divides both expressions is a; b will not divide *both* expressions; the highest power of c which divides both is c^3; and the highest power of d which divides both is d^2. Hence the H. C. F. is ac^3d^2.

Also a^3bc^4, $a^2b^3c^6d$ and $a^4bc^4d^2$ are all divisible by a^2, by b, and by c^4; and therefore the H. C. F. of the three expressions is a^2bc^4.

From the above examples it will be seen that the H. C. F. of two or more simple expressions is obtained by taking *each letter which is common to all the expressions to the lowest power which it has in any one of them.*

If the expressions have numerical coefficients, the G. C. M. of these can be found by arithmetic, and prefixed as a coefficient to the algebraical H. C. F.

EXAMPLES. XXII.

Find the H. C. F. of

1. a^3b^2 and a^2b^3.
2. abc^2 and a^2bc^3.
3. $9ab^3$ and $6a^2b$.
4. $4x^3y$ and $10xy^3$.
5. $24a^3b^3x^4$ and $60a^2b^4x^6$.
6. $a^2b^3x^5$ and $3b^5x$.
7. $9a^2b^3x^4y^5$ and $8x^3y^6$.
8. $\frac{3}{2}a^3b^3c^8$ and $2b^3c^6$.
9. $42axy^2z^3$ and $77b^3y^4$.
10. ab^3, a^2bc and abc^2.
11. $3x^2yz^3$, $15xy^3z^2$ and $10x^2y^2z^2$.
12. $ab^2c^3x^4$, $a^4bc^2x^3$ and $a^3b^2cx^2$.

106. When the factors of two or more *compound* expressions are known, their H. C. F. can be at once written down. The H. C. F. will be the product obtained by taking *each factor which is common to all the expressions to the lowest power which it has in any one of them.*

Consider for example, the expressions $(x-a)^3(x+b)^3$ and $(x-a)^2(x+b)^4$. It is clear that both expressions are divisible by $(x-a)^2$, but by no higher power of $(x-a)$. Also both expressions are divisible by $(x+b)^3$ but by no higher power of $x+b$.

Hence the H.C.F. is $(x-a)^2(x+b)^3$.

Again $a^2b(a-b)^2(a+b)^3$ and $ab^2(a-b)^3(a+b)^4$ are both divisible by a, by b, by $(a-b)^2$, and by $(a+b)^3$. Hence the H.C.F. is $ab(a-b)^2(a+b)^3$.

In the following examples the factors, although not given, can be readily found.

Ex. 1. Find the H.C.F. of $a^4b^2-a^2b^4$ and $a^4b^3+a^3b^4$.

Since $a^4b^2-a^2b^4=a^2b^2(a^2-b^2)=a^2b^2(a-b)(a+b)$,
and $a^4b^3+a^3b^4=a^3b^3(a+b)$;
we see that the H.C.F. is $a^2b^2(a+b)$.

Ex. 2. Find the H.C.F. of $a^3+3a^2b+2ab^2$ and $a^4+4a^3b+3a^2b^2$.

$a^3+3a^2b+2ab^2=a(a^2+3ab+2b^2)=a(a+b)(a+2b)$,
and $a^4+4a^3b+3a^2b^2=a^2(a^2+4ab+3b^2)=a^2(a+b)(a+3b)$;
hence the H.C.F. is $a(a+b)$.

Ex. 3. Find the H.C.F. of $3a^3+2a^2-a$ and $5a^4+3a^3-2a^2$.

$3a^3+2a^2-a=a(3a^2+2a-1)=a(3a-1)(a+1)$,
and $5a^4+3a^3-2a^2=a^2(5a^2+3a-2)=a^2(5a-2)(a+1)$;
hence the H.C.F. is $a(a+1)$.

EXAMPLES. XXIII.

Find the H.C.F. of

1. $x^2(x-a)^3$ and $x^3(x-a)^2$. 2. a^2-b^2 and $(a+b)^2$.
3. $3ab(a+b)^2$ and $2(a-b)(a+b)^3$. 4. $b^4c^6(b+c)^2$ and $b^6c^3(b+c)^3$.
5. $a^3b^3+ab^5$ and $a^6-a^2b^4$. 6. a^2b+3b^3 and $a^6-9a^2b^4$.
7. $a^2x^3+2a^3x^2$ and $a^2x^4-4a^4x^2$.
8. $a^6x^2-4a^4x^4$ and $a^6x^2-16a^2x^6$. 9. x^2+3x+2 and x^2+6x+8.
10. $x^3+3x^2y+2xy^2$ and $x^4+6x^3y+8x^2y^2$.
11. $3x^2-4x+1$ and $4x^2-5x+1$.
12. $3a^2-4ab+b^2$ and $4a^4-5a^3b+a^2b^2$.

107. Although we cannot, in general, find the factors of a compound expression of higher degree than the second [Art. 103], we can always find the highest *common* factor of any *two* expressions by the following process, which will be seen to be analogous to that used in arithmetic to find the greatest common measure of two numbers.

RULE. Arrange the two expressions in descending powers of some common letter, and divide the expression which is of the highest degree in the common letter by the other: if both expressions are of the same degree it is immaterial which is used as the divisor. Take the remainder, if any, after the first division for a new divisor, and the former divisor as dividend; and continue the process until there is no remainder. The last divisor will be the H. C. F. of the two given expressions.

For example, to find the H. C. F. of x^3+x^2-2 and x^3+2x^2-3: the process is as follows

$$\begin{array}{rrrl}
x^3+x^2-2\,) & x^3+2x^2-3\,(\,1 & & \\
 & \underline{x^3+x^2-2} & & \\
 & x^2-1\,) & x^3+x^2-2\,(\,x+1 & \\
 & & \underline{x^3-x\phantom{{}+x^2-2}} & \\
 & & x^2+x-2 & \\
 & & \underline{x^2\phantom{{}+x}-1} & \\
 & & x-1\,) & x^2-1\,(\,x+1 \\
 & & & \underline{x^2-x} \\
 & & & \phantom{x^2-{}}x-1 \\
 & & & \phantom{x^2-{}}\underline{x-1}
\end{array}$$

Thus the H. C. F. is $x-1$.

108. It is only *compound* factors which are to be found by the above rule; the H. C. F. of any *simple* factors of the

given expressions must be found by inspection. For example, to find the H. C. F. of

$$a^2x^4 + a^2x^3 - 2a^2x \text{ and } abx^5 + 2abx^4 - 3abx^2.$$

Since $a^2x^4 + a^2x^3 - 2a^2x = a^2x(x^3 + x^2 - 2)$,

and $abx^5 + 2abx^4 - 3abx^2 = abx^2(x^3 + 2x^2 - 3)$,

it is clear that the H. C. F. of the simple factors is ax. We have already found that the H. C. F. of $x^3 + x^2 - 2$ and $x^3 + 2x^2 - 3$ is $x - 1$.

Hence the whole H. C. F. is $ax(x - 1)$.

109. The rule given in Art. 107 for finding the H. C. F. of two expressions which have no simple factors, may be proved as follows.

Let A and B stand for the two expressions, which are supposed to be arranged according to descending powers of some common letter, and let A be of not higher dimensions than B in the common letter.

Let B be divided by A, and let the quotient be Q and the remainder R, so that the first stage of the process of finding the H. C. F. of A and B is as under

$$\begin{array}{r} A\,)\;\; B\;\;(\,Q \\ \underline{AQ}\;\;\;\;\, \\ R\;\;\;\;\; \end{array}$$

Then we have

$$B = AQ + R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

and $$R = B - AQ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

Now any factor of both A and R will clearly be a factor of $AQ + R$, that is, from (i), will be a factor of B.

Thus any common factor of A and R is also a common factor of A and B.

Again, any factor of both B and A will clearly be a factor of $B-AQ$; that is, from (ii), will be a factor of R.

Thus any common factor of A and B is also a common factor of A and R.

It follows therefore that the common factors of A and B are *exactly the same* as the common factors of A and R. Hence *the* H.C.F. *of A and B, is the* H.C.F. *of A and R.*

If now we divide A by R, and the remainder is S, the H.C.F. of S and R will similarly be the same as the H.C.F. of A and R, and therefore will be the H.C.F. required. And so on; so that *the* H.C.F. *of any divisor and the corresponding dividend is the* H.C.F. *required.*

But if at any stage there is no remainder, the divisor must be a factor of the corresponding dividend, and that divisor is clearly the H.C.F. of itself and the corresponding dividend. It must therefore be the H.C.F. required.

It should be remarked that by the nature of division the remainders are successively of lower and lower dimensions; and hence, unless the division leaves no remainder at some stage, we must at last come to a remainder which does not contain the common letter, in which case the given expressions have no H.C.F.

We have already remarked that the process we are considering is only to be used to find the H.C.F. of the *compound* factors of the given expressions; and, bearing this in mind, it is clear that any of the expressions which occur may be divided or multiplied by any *simple* quantity without destroying the validity of the process; for the *compound* factors will not be altered by such division or multiplication.

110. Ex. 1. Find the H.C.F. of $x^3+4x^2-8x+24$ and x^4-x^3+8x-8.

Neither expression has any simple factors; we therefore proceed as in Art. 107:

$$\begin{array}{rrrrrrl}
x^3+4x^2-8x+24\ ) & x^4 & -\ x^3 & & +\ 8x & -\ \ 8 & (\ x-5 \\
 & x^4 & +4x^3 & -\ 8x^2 & +24x & & \\
\cline{2-6}
 & & -5x^3 & +\ 8x^2 & -16x & -8 & \\
 & & -5x^3 & -20x^2 & +40x & -120 & \\
\cline{3-6}
 & & & 28x^2 & -56x & +112. &
\end{array}$$

The remainder $28x^2-56x+112=28(x^2-2x+4)$; and since the numerical factor 28 is clearly not a factor of the given expressions, we reject it, and continue the process with x^2-2x+4 as the new divisor:

$$\begin{array}{rrrrrl}
x^2-2x+4\ ) & x^3 & +4x^2 & -\ 8x & +24 & (\ x+6 \\
 & x^3 & -2x^2 & +\ 4x & & \\
\cline{2-5}
 & & 6x^2 & -12x & +24 & \\
 & & 6x^2 & -12x & +24 & \\
\cline{3-5}
\end{array}$$

Thus the H.C.F. is x^2-2x+4.

Ex. 2. Find the H.C.F. of $x^3-4a^2x+15a^3$

and $x^4+a^2x^2+25a^4$.

$$\begin{array}{rrrrrl}
x^3-4a^2x+15a^3\ ) & x^4 & +\ \ a^2x^2 & & +25a^4 & (\ x \\
 & x^4 & -4a^2x^2 & +15a^3x & & \\
\cline{2-5}
 & & 5a^2x^2 & -15a^3x & +25a^4 &
\end{array}$$

Now $5a^2x^2-15a^3x+25a^4=5a^2(x^2-3ax+5a^2)$.

We reject the simple factor $5a^2$, and continue the process with $x^2-3ax+5a^2$ as the new divisor:

$$\begin{array}{r}
x^2-3ax+5a^2\;)\;x^3-4a^2x+15a^3\;(\;x+3a\\
\underline{x^3-3ax^2+\;\;5a^2x\qquad\quad}\\
3ax^2-\;\;9a^2x+15a^3\\
\underline{3ax^2-\;\;9a^2x+15a^3}
\end{array}$$

Hence $x^2-3ax+5a^2$ is the required H.C.F.

Ex. 3. Find the H.C.F. of x^5-y^5 and x^7-y^7.

$$\begin{array}{r}
x^5-y^5\;)\;x^7-y^7\;(\;x^2\\
\underline{x^7-x^2y^5\qquad}\\
x^2y^5-y^7
\end{array}$$

Before using the remainder as a divisor, we reject the *simple* factor y^5; we then have x^2-y^2.

$$\begin{array}{r}
x^2-y^2\;)\;x^5-y^5\;(\;x^3+xy^2\\
\underline{x^5-x^3y^2\qquad\qquad}\\
x^3y^2-y^5\qquad\\
\underline{x^3y^2-xy^4\qquad}\\
xy^4-y^5
\end{array}$$

We reject the simple factor y^4 of the remainder.

$$\begin{array}{r}
x-y\;)\;x^2-y^2\;(\;x+y\\
\underline{x^2-xy\qquad}\\
xy-y^2\\
\underline{xy-y^2}
\end{array}$$

Hence $x-y$ is the required H.C.F.

Ex. 4. Find the H.C.F. of $2x^2-5x+2$ and

$$x^3+4x^2-4x-16.$$

To avoid the inconvenience of fractions, we multiply $x^3+4x^2-4x-16$ by 2. We have shewn in the preceding Article that we may do this, as no additional common factors can be thereby introduced. It will be seen that we multiply a second time in the course of the work, which is generally written down in the following form:

$$\begin{array}{rrrrrl}
2x^2-5x+2\;) & x^3+ & 4x^2- & 4x- & 16 & \\
 & & & & 2 & \\
\hline
 & 2x^3+ & 8x^2- & 8x- & 32\;(& x \\
 & 2x^3- & 5x^2+ & 2x & & \\
\hline
 & & 13x^2- & 10x- & 32 & \\
 & & & & 2 & \\
\hline
 & & 26x^2- & 20x- & 64\;(& 13 \\
 & & 26x^2- & 65x+ & 26 & \\
\hline
 & & 45 & \big|\,45x- & 90 & \\
\cline{4-5}
 & & & x- & 2\;) & 2x^2-5x+2\;(\;2x-1 \\
 & & & & & 2x^2-4x \\
\cline{6-6}
 & & & & & \quad -\;x+2 \\
 & & & & & \quad -\;x+2 \\
\cline{6-6}
\end{array}$$

Thus $x-2$ is the H.C.F.

111. The H.C.F. of more than two expressions is sometimes required, when the factors cannot be determined by inspection.

Now it is clear that any factor which is common to each of three or more expressions, must be a factor of the H.C.F. of any two of them. Hence we first find the H.C.F. of two of the given expressions, and then find the H.C.F. of this result and of the third expression, and so on.

Ex. Find the H.C.F. of x^3+x^2-x-1,

x^3+3x^2-x-3 and x^3+x^2-2.

The H.C.F. of the first two expressions is x^2-1. The H.C.F. of x^2-1 and the third of the given expressions is $x-1$. Hence $x-1$ is the H.C.F. required.

112. The *highest common factor* of two expressions is sometimes called their *greatest common measure* [G. C. M.].

The inappropriateness of the term *greatest* common measure can be seen by an example. If a^2 is a factor of two expressions, so also is a, and a^2 is of *higher dimensions* than a; but, as a may represent any number whatever, a^2 is not necessarily *greater* than a; in fact, if a is positive and less than unity, a^2 is *less* than a.

It should also be noticed that if we give particular numerical values to the letters involved in any two expressions, and in their H. C. F., the numerical value of the H. C. F. is by no means necessarily the G. C. M. of the values of the expressions. This is not the case even when the given expressions and the H. C. F. are all *integral* for the particular values chosen. For example, $a+1$ and a^2+1 have no *algebraical factors* in common; but if we suppose a to be 3, the numerical values of the expressions are 4 and 10, which have 2 as a common factor.

EXAMPLES. XXIV.

Find the H. C. F. of

1. x^2-5x+4 and x^3-5x^2+4.
2. $x^2-5xy+4y^2$ and $x^4-5x^3y+4xy^3$.
3. $2x^2-5x+2$ and $4x^3+12x^2-x-3$.
4. $2x^2-5xy+2y^2$ and $4x^3+12x^2y-xy^2-3y^3$.
5. x^4+3x^2-10 and x^4-3x^2+2.
6. $x^6+3x^4y-10x^2y^2$ and $x^4-3x^2y+2y^2$.
7. $2a^2-5a+2$ and $2a^3-3a^2-8a+12$.
8. $2b^2-5b+2$ and $12b^3-8b^2-3b+2$.
9. x^2-3x+2 and x^3-3x+2.
10. $x^4y^4-3x^2y^2+2$ and $x^6y^6-3x^2y^2+2$.
11. $x^3-3a^2x-2a^3$ and $x^3-ax^2-4a^3$.
12. $2a^3+3a^2b-b^3$ and $4a^3+ab^2-b^3$.
13. a^3+b^3 and $a^4+a^2b^2+b^4$.
14. $8a^3+1$ and $16a^4+4a^2+1$.

15. $x^3+2x^2y-xy^2-2y^3$ and $x^3-2x^2y-xy^2+2y^3$.

16. $2x^3+5x^2+x-8$ and $3x^3-4x^2+9x-8$.

17. $3x^3+x^2+x-2$ and $2x^3-x^2-x-3$.

18. $x^3-4x+15$ and x^4+x^2+25.

19. $3x^2-38x+119$ and $x^3-19x^2+119x-245$.

20. $3x^3-3x^2y+xy^2-y^3$ and $4x^2y-5xy^2+y^3$.

21. $12x^2-15xy+3y^2$ and $6x^3-6x^2y+2xy^2-2y^3$.

22. $2x^2-14x+20$ and $4x(x^2+5)-25(x+1)(x-1)$.

23. x^4+4x^2+16 and $2x^4-x^3+16x-8$.

24. $a^2-4x^2+12x-9$ and $a^2+2a-4x^2+8x-3$.

25. $x^3-3x^2+7x-21$ and $2x^4+19x^2+35$.

26. $2x^4+9x^3+14x+3$ and $3x^4+14x^3+9x+2$.

27. x^3+3x^2-x-3 and $x^4+4x^3-12x-9$.

28. $3x^4+5x^3-7x^2+2x+2$ and $2x^4+3x^3-2x^2+12x+5$.

29. y^3-2y^2+3y-6 and $y^4-y^3-y^2-2y$.

30. $y^4-3yz^3+20z^4$ and $5y^4-3y^3z+64z^4$.

31. $2x^3-11x^2+11x+4$ and $2x^4-3x^3+7x^2-12x-4$.

32. $2x^4+4x^3+3x^2-2x-2$ and $3x^4+6x^3+7x^2+2x+2$.

33. $x^5+2x^4-x^3-3x^2+1$ and $x^6+2x^5-5x^4-x^2+2x+1$.

34. $2x^4-6x^3+3x^2-3x+1$ and $x^7-3x^6+x^5-4x^2+12x-4$.

35. $x^3+(m-3)x^2-m(2m+3)x+6m^2$
and $x^3+(5m-3)x^2+3m(2m-5)x-18m^2$.

36. $mn(x^2+y^2)+xy(m^2+n^2)$ and $mn(x^3+y^3)+xy(m^2y+n^2x)$.

CHAPTER XII.

LOWEST COMMON MULTIPLE.

113. A *common multiple* of two or more algebraical expressions, is an expression which is exactly divisible by each of them.

The *lowest common multiple* of two or more algebraical expressions, is the expression of *lowest dimensions* which is exactly divisible by each of them.

Instead of *lowest common multiple* it is usual to write L. C. M.

We proceed to shew how to find the L.C.M. of given expressions.

114. We first consider *simple* expressions the factors of which can be seen by inspection.

Take, for example, a^3b^2c and $a^2b^3c^4$. The highest power of a which occurs in either expression is a^3; hence any common multiple of the given expressions must contain a^3 as a factor. Any common multiple must also contain b^3 as a factor, and it must also contain c^4 as a factor. *Any* common multiple must therefore contain $a^3b^3c^4$ as a factor; and hence the common multiple of lowest dimensions must be $a^3b^3c^4$.

Again, to find the L.C.M. of a^3bc^4, $a^2b^3c^6d$ and $a^4bc^3d^2$, the highest power of a in the given expressions is a^4; hence any common multiple of them must contain a^4 as a factor. Any common multiple must also contain b^3, c^6 and d^2 as factors. Any common multiple must therefore contain $a^4b^3c^6d^2$ as a factor; hence the L. C. M. required is $a^4b^3c^6d^2$.

From the above examples it will be seen that the L.C.M. of two or more simple expressions is obtained by taking *every letter which occurs in the different expressions to the highest power which it has in any one of them.*

If the expressions have numerical coefficients, the L.C.M. of these can be found by arithmetic, and prefixed as a coefficient to the algebraical L. C. M.

EXAMPLES. XXV.

Find the L. C. M. of

1. a^3b^2 and a^2b^3.
2. abc^2 and a^2bc^3.
3. $9ab^3$ and $6a^2b$.
4. $4x^3y$ and $10xy^3$.
5. $24a^3b^3x^4$ and $60a^2b^4x^6$.
6. $a^2b^3x^5$ and $3b^5x$.
7. $9a^2b^3x^4y^5$ and $8x^3y^6$.
8. $\frac{3}{2}a^3b^3c^8$ and $2b^3c^6$.
9. $42axy^2z^3$ and $77b^3y^4$.
10. ab^3, a^2bc and abc^2.
11. $3x^2yz^3$, $15xy^3z^2$ and $10x^2y^2z^2$.
12. $ab^2c^3x^4$, $a^4bc^2x^3$ and $a^3b^2cx^2$.

115. When the factors of two or more *compound* expressions are known, their L. C. M. can be at once written down.

The L. C. M. will be the product obtained by taking every *factor which occurs in the different expressions to the highest power which it has in any one of them.*

Consider, for example, the expressions $(x-a)(x-b)^2(x-c)^3$ and $(x-a)^4(x-b)(x-c)$.

It is clear that any common multiple must contain $(x-a)^4$ as a factor; it must also contain $(x-b)^2$ and $(x-c)^3$ as factors. Any common multiple must therefore contain

$(x-a)^4(x-b)^2(x-c)^3$ as a factor; and hence the common multiple of lowest dimensions must be $(x-a)^4(x-b)^2(x-c)^3$.

Ex. 1. Find the L.C.M. of $a^4b^2-a^2b^4$ and $a^4b^3+a^3b^4$.

Since $a^4b^2-a^2b^4=a^2b^2(a-b)(a+b)$,

and $a^4b^3+a^3b^4=a^3b^3(a+b)$;

we see that the L.C.M. is $a^3b^3(a-b)(a+b)$.

Ex. 2. Find the L.C.M. of $a^3+3a^2b+2ab^2$ and $a^4+4a^3b+3a^2b^2$.

$$a^3+3a^2b+2ab^2=a(a+b)(a+2b),$$

and $a^4+4a^3b+3a^2b^2=a^2(a+b)(a+3b)$;

hence the L.C.M. is $a^2(a+b)(a+2b)(a+3b)$.

We leave the L.C.M. in the above form, as it is generally most convenient to have it expressed in factors.

Ex. 3. Find the L.C.M. of x^2+2x+1, x^2-2x-3 and x^2+4x+3.

$$x^2+2x+1=(x+1)^2,$$
$$x^2-2x-3=(x+1)(x-3),$$

and $x^2+4x+3=(x+1)(x+3)$;

hence the L.C.M. is $(x+1)^2(x-3)(x+3)$.

EXAMPLES. XXVI.

Find the L.C.M. of

1. $(a-x)(a-2x)$ and $(a-2x)(a-3x)$.
2. $ax^3(a-x)(a-2x)$ and $a^3x(a-2x)(a-3x)$.
3. a^2-b^2 and $(a+b)^2$.
4. $6ab(a+b)^2$ and $4a^2(a^2-b^2)$.
5. x^2+3x+2 and x^2+5x+4.
6. $x^2+3xy+2y^2$ and $x^2+5xy+4y^2$.
7. x^2-4x+3 and x^2-5x+6.
8. $3x^2-4xy^2+y^4$ and $6x^2-5xy^2+y^4$.
9. $(a+b)^2$, $(a-b)^2$ and a^2-b^2.
10. $(x+2y)^2$, $(x-2y)^2$ and x^2-4y^2.
11. $x^2+7x+12$, x^2+6x+8 and x^2+5x+6.
12. $x^2-7xy+12y^2$, $x^2-6xy+8y^2$ and $x^2-5xy+6y^2$.

116. When the factors of the expressions whose L.C.M. is required cannot be seen by inspection we must use the rule for finding the H.C.F. given in the preceding chapter.

Thus, to find the L.C.M. of x^3+x^2-2 and x^3+2x^2-3.

The H.C.F. of the given expressions is $x-1$, and we find by division that

$$x^3+x^2-2=(x-1)(x^2+2x+2),$$

and $$x^3+2x^2-3=(x-1)(x^2+3x+3).$$

Then, since x^2+2x+2 and x^2+3x+3 have no common factors, the L.C.M. required is

$$(x-1)(x^2+2x+2)(x^2+3x+3).$$

117. Let A and B stand for any two algebraical expressions, and let H stand for their H.C.F., and L for their L.C.M.

Let a and b be the quotients when A and B respectively are divided by H; so that

$$A=H\times a,$$

and $$B=H\times b.$$

Since H is the highest common factor of A and B, a and b can have no common factors. Hence the L.C.M. of A and B must be $H\times a\times b$. Thus

$$L=H\,.\,a\,.\,b\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

It follows from (i) that

$$L=Ha\times\frac{Hb}{H}=A\times\frac{B}{H}\ \ldots\ldots\ldots\ldots\text{(ii)},$$

and also that

$$L\times H=Ha\times Hb=A\times B\ \ldots\ldots\ldots\ldots\text{(iii)}.$$

From (ii) we see that *the* L.C.M. *of any two algebraical expressions is found by dividing one of the expressions by their* H.C.F., *and multiplying the quotient by the other expression.*

From (iii) we see that *the product of any two expressions is equal to the product of their* H.C.F. *and* L.C.M.

118. To find the L. C. M. of more than two expressions, whose factors cannot be determined by inspection, we first find the L.C.M. of two of the given expressions, and find the L. C. M. of this result and of the third expression, and so on.

EXAMPLES. XXVII.

Find the L. C. M. of

1. $3a^2bc$, $27a^3b^2c^2$ and $6ab^2c$. 2. a^4b^2, b^3c and a^2c^3.

3. $4a^3$, $6a^3b$ and $8ab^3$. 4. $2a^2$, $6ab^3$ and $4a^3b^2$.

5. $x^3(x-y)^2$, $y^3(x+y)^2$ and $xy(x^2-y^2)$.

6. a^3+a^2b, $a(a-b)$ and a^2-b^2. 7. xy^2-y^2, $x^2y^2+xy^2$ and x^2y-y.

8. $2axy(x-y)$, $3ax^2(x^2-y^2)$ and $4y^2(x+y)^2$.

9. $6(x^2-9)$, $9(x+3)$, $15(x-4)$ and $10(x^2-x-12)$.

10. $4a+4b$, $6a^2-24b^2$ and $a^2-3ab+2b^2$.

11. $4ab^3+4b^3d$, $a^2b^2-b^2d^2$ and $8a^2bd-8abd^2$.

12. $x^2-10x+24$, $x^2-8x+12$ and x^2-6x+8.

13. $x^2-9x-10$, $x^2-7x-30$ and x^2+4x+3.

14. $2x^2-8$, $3x^2-9x+6$ and $6x^2+18x+12$.

15. x^2-3x+2, $2x^2-x-6$ and $3x^2-2x-1$.

16. $6x^2+x-2$, $21x^2+17x+2$ and $14x^2-5x-1$.

17. $3x^2-10xy+3y^2$, $3x^2-4xy+y^2$ and $x^2-4xy+3y^2$.

18. x^3+2x^2-3x and $2x^3+5x^2-3x$.

19. $x^2y^2-9y^2$, $x^2y-yx-6y$ and x^3+x^2-6x.

20. x^3-a^3, x^3+a^3 and $x^4+a^2x^2+a^4$.

21. x^2-1, x^3+x^2+x+1 and x^3-x^2+x-1.

22. $9x^3-x-2$ and $3x^3-10x^2-7x-4$.

23. $x^3-ax^2-a^2x+a^3$ and $x^3+ax^2-a^2x-a^3$.

24. x^3+x^2-4x-4 and $x^3+6x^2+11x+6$.

25. x^4-x^3+8x-8 and $x^3+4x^2-8x+24$.

26. $a^3+6a^2+11a+6$ and $a^3+10a^2+29a+20$.

CHAPTER XIII.

FRACTIONS.

119. To obtain the arithmetical fraction $\frac{5}{7}$, we must divide the unit into 7 equal parts and take 5 of those parts. So also to obtain the fraction $\frac{a}{b}$, *where a and b are positive integers*, we must divide the unit into b equal parts and take a of those parts.

120. We do not stop to consider the properties of the fraction $\frac{a}{b}$, where a and b are restricted to positive integral values, for the letters in Algebra are not so restricted, except in a few special cases. Moreover nothing could be added to the proofs given in Arithmetic; by the employment of letters we could only gain in conciseness of statement.

Thus the fact that a fraction is not altered in value by multiplying the numerator and denominator by the same integer, must be proved precisely as in arithmetic: expressed by means of letters, each of which stands for any positive integral number, the theorem is

$$\frac{a}{b}=\frac{am}{bm}.$$

121. The numerator and the denominator of a fraction, defined as in Art. 119, must both be positive integers: we cannot, for example, with that definition, have such a fraction as $\frac{\frac{3}{4}}{-\frac{2}{5}}$; for to say that the unit is to be divided into $-\frac{2}{5}$ equal parts and that $\frac{3}{4}$ of such parts are to be taken, is without meaning.

We must therefore suppose the letters in $\frac{a}{b}$ to be restricted to positive integral values, or we must alter the definition of $\frac{a}{b}$; and as we cannot restrict the values of the letters, we must entirely dispense with the fractional form, or make some modification in its meaning.

Now, with the definition of Art. 119, to multiply the fraction $\frac{a}{b}$ by b, we must take each of the a parts b times; we thus get ab parts, the parts being such that each b of them make up a unit, and therefore the whole ab parts will make up a units. Thus

$$\frac{a}{b} \times b = a \text{.........................(i).}$$

Dividing each side by b, we have

$$\frac{a}{b} = a \div b \text{ ....(ii).}$$

Now we may *define* the fraction $\frac{a}{b}$ as that quantity which when multiplied by b becomes a; for, as we have just seen, this new definition agrees with that of Art. 119 whenever the definition of Art. 119 has meaning; and by taking this new definition we do away with the necessity of ascribing only positive integral values to the letters.

We may similarly *define* the fraction $\frac{a}{b}$ as the quotient obtained by dividing a by b.

Hence, instead of the definition of Art. 119, which is inapplicable to an algebraical fraction, we have either of the following equivalent definitions.

I. *The algebraical fraction* $\frac{a}{b}$, *where* a *and* b *are supposed to have any values whatever, is that quantity which, when multiplied by* b, *becomes equal to* a.

II. *The algebraical fraction* $\frac{a}{b}$ *is the quotient obtained by dividing* a *by* b.

We have already used the fractional form $\frac{a}{b}$ with the meaning $a \div b$.

122. We now proceed to consider the properties of algebraical fractions; and we shall find that algebraical fractions are added, subtracted, multiplied, divided, and simplified, in precisely the same way as arithmetical fractions.

123. We first shew that the value of a fraction is not altered by multiplying its numerator and denominator by the same quantity.

We have to prove that

$$\frac{a}{b} = \frac{am}{bm},$$

for *all* values of a, b and m.

Let $$x = \frac{a}{b}.$$

Then $$x \times b = \frac{a}{b} \times b.$$

Hence, by the definition of $\frac{a}{b}$,

$$xb = a;$$

$$\therefore\ xbm = am.$$

Divide by bm, and we have

$$x = am \div bm = \frac{am}{bm}.$$

Thus *the value of a fraction is not altered by multiplying its numerator and denominator by the same quantity.*

124. Since by the last Article the value of a fraction is not altered by *multiplying* both the numerator and the denominator by the same quantity, it follows conversely that the value of a fraction is not altered by *dividing* both the numerator and the denominator by the same quantity.

Hence a fraction may be simplified by the rejection of any factor which is common to its numerator and denominator. For example, the fraction $\frac{a^2y}{x^2y}$ takes the simpler form $\frac{a^2}{x^2}$, when the factor y, which is common to its numerator and denominator, is rejected.

When the numerator and denominator of a fraction have no common factors, the fraction is said to be in its *lowest terms.*

To reduce a fraction to its lowest terms, we must divide its numerator and denominator by their H. C. F.: for we thus obtain an equivalent fraction whose numerator and denominator have no common factors.

Ex. 1. Reduce $\frac{3ax^2y}{6a^2xy}$ to its lowest terms.

The H.C.F. of the numerator and denominator is $3axy$; and

$$\frac{3ax^2y}{6a^2xy} = \frac{3ax^2y \div 3axy}{6a^2xy \div 3axy} = \frac{x}{2a}.$$

Ex. 2. Reduce $\frac{a^2b^3}{a^5b^4}$ to its lowest terms.

The H.C.F. of the numerator and denominator is a^2b^3; and

$$\frac{a^2b^3}{a^5b^4} = \frac{a^2b^3 \div a^2b^3}{a^5b^4 \div a^2b^3} = \frac{1}{a^3b}.$$

Ex. 3. Reduce $\frac{2a^3b^2xy^4}{3ab^3x^3y^2}$ to its lowest terms.

The H.C.F. of the numerator and denominator is ab^2xy^2; and

$$\frac{2a^3b^2xy^4}{3ab^3x^3y^2} = \frac{2a^3b^2xy^4 \div ab^2xy^2}{3ab^3x^3y^2 \div ab^2xy^2}$$
$$= \frac{2a^2y^2}{3bx^2}.$$

125. Instead of reducing a fraction to its lowest terms by dividing the numerator and denominator by their H.C.F., we may divide by *any* common factor, and repeat the process until the fraction is reduced to its lowest terms.

Thus $$\frac{a^4b^3c^3}{a^2b^4c^4} = \frac{a^2b^3c^3}{b^4c^4} = \frac{a^2c^3}{bc^4} = \frac{a^2}{bc}.$$

The above process may be written down more compactly as follows:

$$\frac{\overset{a^2}{\cancel{a^4}}\cancel{b^3}\cancel{c^3}}{\cancel{a^2}\underset{b}{\cancel{b^4}}\underset{c}{\cancel{c^4}}} = \frac{a^2}{bc}.$$

EXAMPLES. XXVIII.

Reduce the following fractions to their simplest forms:

1. $\dfrac{a^2b}{ab^2}$.

2. $\dfrac{x^3y^2}{x^2y^3}$.

3. $\dfrac{2a^2bc^2}{4a^4b^3c}$.

4. $\dfrac{6a^5b^3c^4}{4a^7b^5c^4}$.

5. $\dfrac{x^4y^6z^2}{x^6y^2z^2}$.

6. $\dfrac{12x^7y^2z^{10}}{16x^2y^2z^2}$.

7. $\dfrac{15a^3b^2c^7x^5}{25a^2b^4c^3x^8}$.

8. $\dfrac{125ab^2c^3d^4}{150a^4b^3c^2d}$.

9. $\dfrac{3a^5x^3yz^4}{5ab^4xy^3z}$.

10. $\dfrac{5a^3b^4c^5xy^2}{7b^4c^3xy^3}$.

11. $\dfrac{14ab^2c^5xy^2z^5}{21a^5b^2cx^5y^2z}$.

12. $\dfrac{3a^3bc^4x^5y^2z}{4a^2b^3cx^3y^4z^3}$.

126. When the numerator and denominator of a fraction are compound expressions whose factors can be seen by inspection, write the numerator and denominator as the product of factors of the lowest possible dimensions; the factors which are common to the numerator and denominator will then be obvious, and can be removed.

Ex. 1. Simplify $\dfrac{a^2-ax}{a^2-x^2}$.

$$\frac{a^2-ax}{a^2-x^2}=\frac{a(a-x)}{(a-x)(a+x)}$$
$$=\frac{a}{a+x}.$$

Ex. 2. Simplify $\dfrac{x^4-x^2}{x^4-1}$.

$$\frac{x^4-x^2}{x^4-1}=\frac{x^2(x^2-1)}{(x^2-1)(x^2+1)}$$
$$=\frac{x^2}{x^2+1}.$$

Ex. 3. Simplify $\dfrac{x^2-7x+10}{x^2-5x+6}$.

$$\frac{x^2-7x+10}{x^2-5x+6}=\frac{(x-5)(x-2)}{(x-3)(x-2)}=\frac{x-5}{x-3}.$$

Ex. 4. Simplify $\dfrac{x^2-ax}{a^2-x^2}$.

$$\frac{x^2-ax}{a^2-x^2}=\frac{x(x-a)}{(a-x)(a+x)}.$$

Now $x-a=-(a-x)$; hence dividing the numerator and denominator by $a-x$, we have the equivalent fraction

$$\frac{-x}{a+x}.$$

If we divide the numerator and denominator by $x-a$, we have

$$\frac{x}{-(a+x)}.$$

By the Law of Signs in Division $\dfrac{-x}{a+x}$ and $\dfrac{x}{-(a+x)}$ are both equal to $-\dfrac{x}{a+x}$; and this latter is the form in which the result is usually left.

It should be remarked that the value of a fraction is not altered by changing the signs of *all* the terms in the numerator and also of *all* the terms in the denominator; for this is equivalent to multiplying both numerator and denominator by -1.

EXAMPLES. XXIX.

Simplify

1. $\dfrac{2ab}{a^2+ab}$.

2. $\dfrac{x^2y^2}{x^2-x^2y^2}$.

3. $\dfrac{a^2-ab}{a^2+ab}$.

4. $\dfrac{x^2+ax}{x^2-a^2}$.

5. $\dfrac{(x-1)^2}{x^2-1}$.

6. $\dfrac{x^2-x^2y^2}{(x+xy)^2}$.

7. $\dfrac{x^2+2x}{x^2-4}$.

8. $\dfrac{x^4+x^3}{x^4-1}$.

9. $\dfrac{4x-16}{x^2-16}$.

10. $\dfrac{2x^3-4x^4}{x^2-4x^4}$.

11. $\dfrac{x-2}{4-x^2}$.

12. $\dfrac{a-3}{9-a^2}$.

13. $\dfrac{a^2x^2-x^4}{x^4-a^4}$.

14. $\dfrac{x^4-a^2}{a^2-ax^2}$.

15. $\dfrac{x^5-a^2x^3}{x^4-a^4}$.

16. $\dfrac{a^2x^2y^2-x^4y^4}{x^4y^4-a^4}$.

17. $\dfrac{5ax-15a^2}{x^2-9a^2}$.

18. $\dfrac{3x^2-12ax}{48a^2-3x^2}$.

19. $\dfrac{a^2-2ax+x^2}{a^2-x^2}$.

20. $\dfrac{a^4+2a^2b^2+b^4}{a^4-b^4}$.

21. $\dfrac{x^2-1}{x^3-1}$.

22. $\dfrac{x^4-1}{x^6-1}$.

23. $\dfrac{x^2-5x+6}{x^2-7x+12}$.

24. $\dfrac{1-5a+6a^2}{1-7a+12a^2}$.

25. $\dfrac{x^2-9x+20}{x^2+6x-55}$.

26. $\dfrac{1-9y^2+20y^4}{1+6y^2-55y^4}$.

27. $\dfrac{x^2-8xy+7y^2}{x^2-3xy-28y^2}$.

28. $\dfrac{1-8a^2b^2+7a^4b^4}{1-3a^2b^2-28a^4b^4}$.

29. $\dfrac{(a^3-x^3)(a+x)}{(a^3+x^3)(a-x)}$.

30. $\dfrac{(a^3-b^3)(a^2-ab+b^2)}{(a^3+b^3)(a^2+ab+b^2)}$.

31. $\dfrac{a^4-b^4}{(a^3-b^3)(a+b)}$.

32. $\dfrac{(a^6-b^6)(a-b)}{(a^3-b^3)(a^4-b^4)}$.

127. When the factors of the numerator and denominator of a fraction cannot be found by inspection, their H.C.F. can be found by the rule given in Chapter XI.; and the fraction will be reduced to its simplest form by dividing the numerator and denominator by their H.C.F.

Thus to simplify $\dfrac{x^3-23x+10}{5x^3-23x^2+4}$.

The H.C.F. is x^2-5x+2, and

$$x^3-23x+10=(x^2-5x+2)(x+5),$$
$$5x^3-23x^2+4=(x^2-5x+2)(5x+2).$$

Hence the given fraction is equal to

$$\frac{x+5}{5x+2}.$$

EXAMPLES. XXX.

Simplify

1. $\dfrac{x^3-3x+2}{2x^3-3x^2+1}$.

2. $\dfrac{3x^2-8x+5}{x^3-4x^2+5x-2}$.

3. $\dfrac{x^3+3x^2-20}{x^4-x^2-12}$.

4. $\dfrac{x^3+6x^2+11x+6}{x^3+5x^2+6x}$.

5. $\dfrac{2x^3+ax^2+4a^2x-7a^3}{x^3-7ax^2+8a^2x-2a^3}$.

6. $\dfrac{2x^3+3x^2+4x-3}{6x^3+x^2-1}$.

7. $\dfrac{x^3-3x-2}{x^4+2x^3+2x^2+2x+1}$.

8. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

9. $\dfrac{x^4+2x^3-3x^2-7x-2}{2x^4+x^3-6x^2-5x-1}$.

128. Since the value of a fraction is unaltered by multiplying its numerator and denominator by the same quantity [Art. 123], any number of fractions can be reduced to equivalent fractions all of which have the same denominator.

The process is precisely the same as in Arithmetic, and is as follows.

First find the L.C.M. of all the denominators; then divide the L.C.M. by the denominator of one of the fractions, and multiply the numerator and denominator of that fraction by the quotient; and deal in a similar manner with all the other fractions: we thus obtain fractions, equal to the given fractions, and all of which have the L.C.M. for denominator.

For example, to reduce

$$\frac{x}{a^2b(x+a)}, \quad \frac{y}{ab^2(x-a)} \text{ and } \frac{z}{ab(x^2-a^2)},$$

to a common denominator.

The L.C.M. of the denominators is $a^2b^2(x^2-a^2)$. Dividing the L.C.M. by $a^2b(x+a)$, $ab^2(x-a)$ and $ab(x^2-a^2)$, we have the quotients $b(x-a)$, $a(x+a)$ and ab respectively. Hence

$$\frac{x}{a^2b(x+a)} = \frac{x \times b(x-a)}{a^2b(x+a) \times b(x-a)} = \frac{bx(x-a)}{a^2b^2(x^2-a^2)},$$

$$\frac{y}{ab^2(x-a)} = \frac{y \times a(x+a)}{ab^2(x-a) \times a(x+a)} = \frac{ay(x+a)}{a^2b^2(x^2-a^2)},$$

and

$$\frac{z}{ab(x^2-a^2)} = \frac{z \times ab}{ab(x^2-a^2) \times ab} = \frac{abz}{a^2b^2(x^2-a^2)}.$$

In the above process it is not absolutely necessary to take the *lowest* common multiple of the denominators; *any* common multiple will do equally well, but by using the L.C.M. there is a saving of labour.

129. The sum or difference of two fractions which have the same denominator is a fraction whose numerator

is the sum or difference of their numerators, and which has the common denominator. This follows from Art. 70.

When two fractions have not the same denominator, they must first be reduced, by the method explained in the last Article, to equivalent fractions which have the same denominator: their sum or difference will then be found by taking the sum or difference of their numerators, retaining the common denominator.

Thus $$\frac{a}{x}+\frac{b}{x}=\frac{a+b}{x},$$

$$\frac{a}{x}-\frac{b}{x}=\frac{a-b}{x},$$

$$\frac{a}{x}+\frac{b}{y}=\frac{ay}{xy}+\frac{bx}{xy}=\frac{ay+bx}{xy},$$

and $$a+\frac{b}{y}=\frac{ay}{y}+\frac{b}{y}=\frac{ay+b}{y}.$$

When more than two fractions are to be added, or when there are several fractions some of which are to be added and the others subtracted, the process is precisely the same. The fractions are first reduced to a common denominator, and then the numerators of the reduced fractions are added or subtracted as may be required.

Thus $$\frac{a}{x}+\frac{b}{x}-\frac{c}{x}=\frac{a+b-c}{x},$$

and $$\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=\frac{ayz}{xyz}+\frac{bxz}{xyz}-\frac{cxy}{xyz}=\frac{ayz+bxz-cxy}{xyz}.$$

It may be necessary to remind the student that when there is no sign between a fraction and a letter or number, the sign of *multiplication* is understood. Thus $2\frac{a}{b}$ means $2 \times \frac{a}{b}$.

Ex. 1. Find the value of $\frac{1}{x-a} + \frac{1}{x+a}$.

The L.C.M. of the denominators is $(x-a)(x+a)$; and

$$\frac{1}{x-a} + \frac{1}{x+a} = \frac{x+a}{(x-a)(x+a)} + \frac{x-a}{(x+a)(x-a)}$$

$$= \frac{x+a+(x-a)}{x^2-a^2} = \frac{2x}{x^2-a^2}.$$

Ex. 2. Find the value of $\frac{a}{a-x} + \frac{ax}{x^2-a^2}$.

Beginners should always see that the denominators of the fractions which are to be added are all arranged according to descending powers, or all according to ascending powers, of some particular letter. This is not the case in the present example; but

$$\frac{ax}{x^2-a^2} = \frac{ax}{-(a^2-x^2)} = \frac{-ax}{a^2-x^2}.$$

We then have

$$\frac{a}{a-x} + \frac{-ax}{a^2-x^2} = \frac{a(a+x)}{a^2-x^2} + \frac{-ax}{a^2-x^2}$$

$$= \frac{a(a+x)-ax}{a^2-x^2}$$

$$= \frac{a^2}{a^2-x^2}.$$

Ex. 3. Simplify $\frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4}$.

It is sometimes best not to add all the fractions at once; this is particularly the case when the denominators are not all of the same dimensions.

$$\frac{1}{1-x}+\frac{1}{1+x}=\frac{(1+x)+(1-x)}{(1-x)(1+x)}=\frac{2}{1-x^2};$$

then $$\frac{2}{1-x^2}+\frac{2}{1+x^2}=\frac{2(1+x^2)+2(1-x^2)}{(1-x^2)(1+x^2)}=\frac{4}{1-x^4};$$

and finally

$$\frac{4}{1-x^4}+\frac{4}{1+x^4}=\frac{4(1+x^4)+4(1-x^4)}{(1-x^4)(1+x^4)}=\frac{8}{1-x^8}.$$

Ex. 4. Find the value of $\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}$.

Here again it is best not to reduce all the fractions to a common denominator at once: the work is simplified by proceeding as under.

$$\frac{1}{x-1}-\frac{1}{x+1}=\frac{x+1-(x-1)}{(x-1)(x+1)}=\frac{2}{x^2-1},$$

$$\frac{1}{x-2}-\frac{1}{x+2}=\frac{x+2-(x-2)}{(x-2)(x+2)}=\frac{4}{x^2-4},$$

and $$\frac{2}{x^2-1}+\frac{4}{x^2-4}=\frac{2(x^2-4)+4(x^2-1)}{(x^2-1)(x^2-4)}=\frac{6x^2-12}{x^4-5x^2+4}.$$

Ex. 5. Find the value of $\frac{1}{x^2-5x+6}-\frac{1}{x^2-7x+12}$.

$$\frac{1}{x^2-5x+6}=\frac{1}{(x-2)(x-3)},$$

and $$\frac{1}{x^2-7x+12}=\frac{1}{(x-3)(x-4)};$$

hence the L.C.M. of the denominators is $(x-2)(x-3)(x-4)$. Hence we have

$$\frac{x-4}{(x-2)(x-3)(x-4)}-\frac{x-2}{(x-2)(x-3)(x-4)}$$

$$=\frac{(x-4)-(x-2)}{(x-2)(x-3)(x-4)}=\frac{-2}{(x-2)(x-3)(x-4)}$$

$$=-\frac{2}{(x-2)(x-3)(x-4)}.$$

Ex. 6. Find the value of

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}.$$

In examples of this kind it is best for beginners to arrange all the factors in the denominators of the fractions so that a precedes b or c, and that b precedes c. We therefore change $b-a$ into $-(a-b)$, $c-a$ into $-(a-c)$, and $c-b$ into $-(b-c)$. Then

$$\frac{1}{(b-a)(b-c)}=\frac{1}{-(a-b)(b-c)}=\frac{-1}{(a-b)(b-c)},$$

and

$$\frac{1}{(c-a)(c-b)}=\frac{1}{\{-(a-c)\}\{-(b-c)\}}=\frac{1}{(a-c)(b-c)}.$$

Hence we have to simplify

$$\frac{1}{(a-b)(a-c)}+\frac{-1}{(a-b)(b-c)}+\frac{1}{(a-c)(b-c)}.$$

The L.C.M. is $(a-b)(a-c)(b-c)$; and

$$\frac{1}{(a-b)(a-c)}=\frac{b-c}{(a-b)(a-c)(b-c)},$$

$$\frac{-1}{(a-b)(b-c)}=\frac{-(a-c)}{(a-b)(b-c)(a-c)},$$

and

$$\frac{1}{(a-c)(b-c)}=\frac{a-b}{(a-b)(a-c)(b-c)}.$$

Hence we have

$$\frac{b-c-(a-c)+a-b}{(a-b)(a-c)(b-c)}=\frac{b-c-a+c+a-b}{(a-b)(a-c)(b-c)}$$
$$=\frac{0}{(a-b)(a-c)(b-c)}=0.$$

EXAMPLES. XXXI.

Find the value of

1. $\frac{1}{a-b}-\frac{1}{a+b}$.

2. $\frac{1}{x-3}-\frac{1}{x-2}$.

3. $\frac{1}{x-2}+\frac{2}{(x-2)^2}$.

4. $\frac{x+y}{x-y}-\frac{x-y}{x+y}+\frac{4y^2}{x^2-y^2}$.

5. $\frac{2a}{a+b}+\frac{2b}{a-b}-\frac{a^2+b^2}{a^2-b^2}$.

6. $\frac{a}{a-1}-1-\frac{1}{a(a-1)}$.

7. $\frac{2}{x-2}-\frac{1}{x+2}-\frac{x+6}{x^2+4}$.

8. $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}$.

9. $\frac{3-x}{1-3x}-\frac{3+x}{1+3x}-\frac{1-16x}{9x^2-1}$.

10. $\frac{a}{a-x}+\frac{a}{a+x}+\frac{2a^2}{a^2+x^2}$.

11. $\frac{a}{a-x}+\frac{a}{a+x}+\frac{2a^2}{a^2+x^2}+\frac{4a^4}{a^4+x^4}$.

12. $\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\frac{1}{x+4}$.

13. $\frac{1}{x-1}-\frac{2}{x}+\frac{1}{x+1}$.

14. $\frac{1}{a}-\frac{2}{a+1}+\frac{1}{a+2}$.

15. $\frac{1}{x-3}-\frac{3}{x-1}+\frac{3}{x+1}-\frac{1}{x+3}$.

16. $\frac{1}{a}-\frac{3}{a+1}+\frac{3}{a+2}-\frac{1}{a+3}$.

17. $\frac{1}{x-2}-\frac{4}{x-1}+\frac{6}{x}-\frac{4}{x+1}+\frac{1}{x+2}$.

18. $\frac{1}{a}-\frac{4}{a+1}+\frac{6}{a+2}-\frac{4}{a+3}+\frac{1}{a+4}$.

19. $\dfrac{1}{x^2-5x+6}-\dfrac{2}{x^2-4x+3}+\dfrac{1}{x^2-3x+2}$.

20. $\dfrac{1}{x^2+5ax+6a^2}-\dfrac{2}{x^2+4ax+3a^2}+\dfrac{1}{x^2+3ax+2a^2}$.

21. $\dfrac{x-1}{(x-2)(x-3)}-\dfrac{2(x-2)}{(x-3)(x-1)}+\dfrac{x-3}{(x-1)(x-2)}$.

22. $\dfrac{1}{x(x-1)}+\dfrac{2}{1-x^2}+\dfrac{1}{x(x+1)}$.

23. $\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{x^2-5ax+6a^2}+\dfrac{2}{x-3a}$.

24. $\dfrac{3+2a}{2-a}-\dfrac{2-3a}{2+a}+\dfrac{16a-a^2}{a^2-4}$.

25. $\dfrac{x+ay}{x-ay}-\dfrac{x-ay}{x+ay}+2\dfrac{x^2+a^2y^2}{x^2-a^2y^2}$.

26. $\dfrac{x^2-2x}{x^2-1}-\dfrac{x+3}{x+1}-\dfrac{4x}{1-x}$.

27. $\dfrac{1}{1-x}+\dfrac{1}{x+1}+\dfrac{1+2x-x^2}{1-x^2}$.

28. $\dfrac{1}{x+3y}+\dfrac{6y}{x^2-9y^2}-\dfrac{1}{3y-x}$.

29. $\dfrac{x^2-(y-z)^2}{(x+y)^2-z^2}+\dfrac{y^2-(z-x)^2}{(y+z)^2-x^2}+\dfrac{z^2-(x-y)^2}{(z+x)^2-y^2}$.

30. $\dfrac{a^2-(b+c)^2}{(a+b)^2-c^2}+\dfrac{b^2-(c+a)^2}{(b-c)^2-a^2}+\dfrac{c^2-(a-b)^2}{(c-a)^2-b^2}$.

130. We have now to shew how to multiply any two algebraical fractions.

Let the fractions be $\dfrac{a}{b}$ and $\dfrac{c}{d}$.

Let $$x=\frac{a}{b}\times\frac{c}{d}.$$

Multiply by $b\times d$; then

$$x\times b\times d=\frac{a}{b}\times\frac{c}{d}\times b\times d$$
$$=\frac{a}{b}\times b\times\frac{c}{d}\times d,$$

by the Commutative Law [Art. 31].

But $$\frac{a}{b} \times b = a, \text{ and } \frac{c}{d} \times d = c;$$

$$\therefore\ xbd = ac.$$

Divide by bd; then

$$x = \frac{ac}{bd}.$$

That is $$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Thus *the product of any two algebraical fractions is another fraction whose numerator is the product of their numerators, and whose denominator is the product of their denominators.*

The product of any number of fractions is found by the same rule. For example:

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f}$$

$$= \frac{ace}{bdf}.$$

Again, $$\frac{a}{b} \times \frac{c}{d} \times \frac{d}{a} = \frac{acd}{bda}$$

$$= \frac{c}{b}.$$

And $$\frac{x+1}{x+2} \times \frac{x+2}{x+3} \times \frac{x+3}{x+1} = \frac{(x+1)(x+2)(x+3)}{(x+2)(x+3)(x+1)} = 1.$$

131. Let $$x = \frac{a}{b} \div \frac{c}{d}.$$

Then $$x \times \frac{c}{d} = \frac{a}{b} \div \frac{c}{d} \times \frac{c}{d} = \frac{a}{b};$$

$$\therefore\ x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}.$$

Hence $$x = \frac{a}{b} \times \frac{d}{c},$$

since $$\frac{c}{d} \times \frac{d}{c} = \frac{cd}{dc} = 1.$$

Thus *to divide by any fraction is the same as to multiply by its reciprocal.*

Two fractions which are such that the numerator of each is the denominator of the other are called *reciprocal* fractions. Thus $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$, and $\frac{1}{a}$ is the reciprocal of $\frac{a}{1}$ or a.

As particular cases of multiplication and division, we have

$$\frac{a}{b} \times \frac{c}{1} = \frac{ac}{b},$$

that is $\frac{a}{b} \times c = \frac{ac}{b}$, which may be written

$$a \div b \times c = a \times c \div b.$$

And $$\frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc},$$

that is $\frac{a}{b} \div c = \frac{a}{bc}$, which may be written

$$a \div b \div c = a \div bc.$$ [See Art. 36.]

EXAMPLES. XXXII.

Reduce the following fractions to their simplest forms:

1. $\frac{2a}{3c} \times \frac{3c}{4a}$.

2. $\frac{5a^2}{6bc} \times \frac{3b^2}{10ca}$.

3. $\frac{a}{c} \div \frac{c}{d}$.

4. $\frac{2a^2}{bc} \div \frac{3ab}{c^2}$.

5. $\frac{a}{b} \times \frac{c}{d} \times \frac{bd}{ac}$.

6. $\frac{a^2}{bc} \times \frac{b^2}{ca} \times \frac{c^2}{ab}$.

7. $\frac{b}{a} \times \frac{b}{c} \div \frac{c}{a}$.
8. $\frac{a^2}{b^2} \times \frac{b^2}{c^2} \div \frac{c^2}{a^2}$.

9. $\frac{2a}{bc} \times \frac{2b}{ca} \times \frac{2c}{ab}$.
10. $\frac{2x^3}{3yz} \times \frac{3y^3}{5zx} \times \frac{5z^3}{2xy}$.

11. $\frac{3axy^3}{5b^2} \div \frac{6ay^3}{10b^2x}$.
12. $\frac{5a^2bc}{3b^2c^2a} \div \frac{5a^2bc^2}{3ab^2c^3}$.

13. By what must $\frac{3a^2b}{5x^2y}$ be multiplied, that the product may be $\frac{2a}{5x}$?

14. By what must $\frac{x^2y}{a^2z^2}$ be divided, that the quotient may be $\frac{b^2x}{a^2z}$?

15. $\frac{x^3 - a^3}{x^2 - 4a^2} \times \frac{x + 2a}{x - a}$.
16. $\frac{a^3 - x^3}{a^3 + x^3} \div \frac{(a - x)^2}{a^2 - x^2}$.

17. $\frac{x - 1}{x - 2} \times \frac{x - 2}{x - 3} \times \frac{x - 3}{x - 4}$.
18. $\frac{x + 1}{(x + 2)^2} \times \frac{x + 2}{(x + 3)^2} \times \frac{x + 3}{(x + 1)^2}$

19. $\frac{x^2 - 3x + 2}{x^2 - 5x + 6} \times \frac{x^2 - 7x + 12}{x^2 - 5x + 4}$.
20. $\frac{x^2 - 1}{x^2 + 3x - 10} \times \frac{x^2 - 25}{x^2 - 3x - 4}$.

21. $\frac{a + x}{(a - x)^2} \times \frac{a^2 - x^2}{a^2 + x^2} \times \frac{a^4 - x^4}{(a + x)^3}$.

132. We will now give some examples of more complex fractional expressions.

Ex. 1. Simplify $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

Ex. 2. Simplify $\dfrac{\frac{a + x}{a - x} - \frac{a - x}{a + x}}{\frac{a + x}{a - x} + \frac{a - x}{a + x}}$.

$$\frac{a+x}{a-x}-\frac{a-x}{a+x}=\frac{(a+x)(a+x)-(a-x)(a-x)}{(a-x)(a+x)}=\frac{4ax}{a^2-x^2},$$

and

$$\frac{a+x}{a-x}+\frac{a-x}{a+x}=\frac{(a+x)(a+x)+(a-x)(a-x)}{(a-x)(a+x)}=\frac{2a^2+2x^2}{a^2-x^2}.$$

Hence the given fraction is equal to

$$\frac{4ax}{a^2-x^2}\div\frac{2a^2+2x^2}{a^2-x^2}=\frac{4ax}{a^2-x^2}\times\frac{a^2-x^2}{2a^2+2x^2}=\frac{2ax}{a^2+x^2}.$$

Ex. 3. Simplify $\dfrac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}$.

$$\frac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}=\frac{x}{x-\dfrac{(x+2)x}{(x+2)x-(x+1)}}$$

$$=\frac{x}{x-\dfrac{x^2+2x}{x^2+x-1}}=\frac{x(x^2+x-1)}{x(x^2+x-1)-(x^2+2x)}$$

$$=\frac{x(x^2+x-1)}{x^3-3x}=\frac{x^2+x-1}{x^2-3}.$$

Ex. 4. Simplify $\dfrac{\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}}{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}}$.

$$\frac{a-b}{1+ab}+\frac{b-c}{1+bc}=\frac{(a-b)(1+bc)+(b-c)(1+ab)}{(1+ab)(1+bc)}$$

$$=\frac{a-b+abc-b^2c+b-c+ab^2-abc}{(1+ab)(1+bc)}$$

$$=\frac{a-c+ab^2-cb^2}{(1+ab)(1+bc)},$$

$$1-\frac{(a-b)(b-c)}{(1+ab)(1+bc)}=\frac{(1+ab)(1+bc)-(a-b)(b-c)}{(1+ab)(1+bc)}$$

$$=\frac{1+ab+bc+ab^2c-ab+b^2+ac-bc}{(1+ab)(1+bc)}$$

$$=\frac{1+ac+b^2+ab^2c}{(1+ab)(1+bc)}.$$

Hence the given fraction is equal to

$$\frac{a-c+ab^2-cb^2}{(1+ab)(1+bc)}\div\frac{1+ac+b^2+ab^2c}{(1+ab)(1+bc)}$$

$$=\frac{a-c+ab^2-cb^2}{1+ac+b^2+acb^2}=\frac{(a-c)(1+b^2)}{(1+ac)(1+b^2)}=\frac{a-c}{1+ac}.$$

EXAMPLES. XXXIII.

Simplify

1. $\left\{\frac{x+2a}{a-2x}-\frac{a+2x}{x-2a}\right\}\times\left\{\frac{3}{2a-x}-\frac{1}{a-x}\right\}$.

2. $\left\{\frac{x}{1+x}+\frac{1-x}{x}\right\}\div\left\{\frac{x}{1+x}-\frac{1-x}{x}\right\}$.

3. $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)\left(\frac{x}{y^2}+\frac{y}{x^2}\right)\div\left(\frac{x^2}{y}+\frac{y^2}{x}\right)\left(\frac{x}{y^2}-\frac{y}{x^2}\right)$.

4. $\dfrac{2}{\frac{1}{x}-\frac{1}{y}}+\dfrac{y}{1-\frac{y}{x}}-\dfrac{x}{1-\frac{x}{y}}$.

5. $\dfrac{x}{1+\frac{x}{y}}+\dfrac{y}{1+\frac{y}{x}}-\dfrac{2}{\frac{1}{x}+\frac{1}{y}}$.

6. $\dfrac{\frac{x}{y}+\frac{y}{x}+2}{x+y}+\dfrac{\frac{x}{y}+\frac{y}{x}-2}{x-y}$.

7. $\dfrac{\frac{a+b}{a+2b}+\frac{b}{a}}{\frac{a+b}{a}-\frac{b}{a+2b}}$.

8. $\dfrac{x^2-x+\frac{x-1}{x+1}}{x+\frac{1}{x+1}}$.

9. $\dfrac{\frac{x+2}{2x+3}-\frac{4x+5}{5x+6}}{\frac{2x+3}{3x+4}-\frac{3x+4}{4x+5}}$.

10. $\dfrac{\frac{1+x}{1+x^2}-\frac{1+x^2}{1+x^3}}{\frac{1+x^2}{1+x^3}-\frac{1+x^3}{1+x^4}}$.

11. $\dfrac{1}{x}-\dfrac{1}{x+\dfrac{1}{3x-\dfrac{1}{x}}}$.

12. $x^3-\dfrac{x^2}{1+\dfrac{1-x}{x+\dfrac{1}{x}}}$.

13. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{a+1}{3-a}}}$.

14. $\dfrac{1}{x^2-\dfrac{x^3+1}{x+\dfrac{1}{x-1}}}$.

133. We conclude this chapter by giving examples of equations which contain fractional expressions.

Ex. 1. Solve $\dfrac{x-1}{5}-\dfrac{3x-1}{8}=\dfrac{43-5x}{6}$.

We may multiply both sides of the equation by 120, the L. C. M. of the denominators of the fractions, without destroying the equality: we thus get rid of fractions, and have

$$24(x-1)-15(3x-1)=20(43-5x);$$

$$\therefore\ 24x-24-45x+15=860-100x.$$

Transposing, we have

$$24x-45x+100x=860+24-15;$$

$$\therefore\ 79x=869.$$

Hence $$x=\frac{869}{79}=11.$$

Ex. 2. Solve $\dfrac{1}{x-3}+\dfrac{2}{x+3}=\dfrac{3}{x+5}$.

The L. C. M. of the denominators is $(x-3)(x+3)(x+5)$. Multiplying both sides of the equation by the L. C. M., we get rid of fractions, and have

$$(x+3)(x+5)+2(x-3)(x+5)=3(x-3)(x+3);$$

$$\therefore\ x^2+8x+15+2x^2+4x-30=3x^2-27.$$

By transposition, we have

$$x^2+2x^2-3x^2+8x+4x=-27-15+30;$$

$$\therefore\ 12x=-12,$$

or
$$x=-1.$$

Ex. 3. Solve $\dfrac{1}{x+1}+\dfrac{1}{x+7}=\dfrac{1}{x+3}+\dfrac{1}{x+5}$.

In this case it is best not to multiply at once by the L. C. M. of the denominators: the work is simplified by proceeding as under.

We have
$$\frac{1}{x+1}-\frac{1}{x+3}+\frac{1}{x+7}-\frac{1}{x+5}=0;$$

$$\therefore\ \frac{x+3-(x+1)}{(x+1)(x+3)}+\frac{x+5-(x+7)}{(x+5)(x+7)}=0,$$

that is
$$\frac{2}{(x+1)(x+3)}-\frac{2}{(x+5)(x+7)}=0;$$

$$\therefore\ (x+1)(x+3)=(x+5)(x+7),$$

or
$$x^2+4x+3=x^2+12x+35;$$

$$\therefore\ 4x-12x=35-3;$$

$$\therefore\ -8x=32,$$

or
$$x=-4.$$

Ex. 4. Solve the equation

$$\frac{x-1}{x+1}+\frac{x+5}{x+7}=\frac{x+1}{x+3}+\frac{x+3}{x+5}.$$

We may write the equation in the following form:

$$\frac{x+1-2}{x+1}+\frac{x+7-2}{x+7}=\frac{x+3-2}{x+3}+\frac{x+5-2}{x+5};$$

$$\therefore\quad 1-\frac{2}{x+1}+1-\frac{2}{x+7}=1-\frac{2}{x+3}+1-\frac{2}{x+5};$$

$$\therefore\quad \frac{2}{x+1}+\frac{2}{x+7}=\frac{2}{x+3}+\frac{2}{x+5},$$

which is the same equation as that in Ex. 3.

EXAMPLES. XXXIV.

Solve the following equations:—

1. $\frac{x-2}{5}-\frac{x-3}{4}=\frac{x-7}{10}$.

2. $\frac{3x-1}{11}-2\frac{x-1}{5}=\frac{2-x}{10}$.

3. $\frac{60-x}{14}-\frac{3x-5}{7}=6-\frac{24-3x}{4}$.

4. $\frac{1-3x}{2}+\frac{3x+1}{2}=\frac{2}{1-3x}$.

5. $\frac{1}{x+5}-\frac{1}{x+6}=\frac{1}{x+6}-\frac{1}{x+8}$.

6. $\frac{1}{x+7}-\frac{1}{x+1}=\frac{1}{x+1}-\frac{1}{x+3}$.

7. $\frac{1}{4x+6}+\frac{1}{6x+4}=\frac{2}{2x+3}$.

8. $\frac{1}{3x+9}+\frac{2}{5x+1}=\frac{2}{x+3}$.

9. $\frac{4x}{x+1}-\frac{x}{x-2}=3$.

10. $\frac{1}{x+4}+\frac{2}{x+6}=\frac{3}{x+5}$.

11. $\frac{x+1}{x-1}-\frac{x-3}{x+3}=\frac{8}{x}$.

12. $\frac{3(x-1)}{x+1}+\frac{2(x+1)}{x-1}=5$.

13. $\frac{8x}{x^3-1}=\frac{1}{x-4}-\frac{1}{x+4}$.

14. $\frac{4}{x^2-1}+\frac{1}{x+1}=\frac{1}{1-x}$.

15. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}$.

16. $\frac{1}{x-5}+\frac{3}{2x-6}=\frac{5}{(x-5)(x-3)}$.

17. $\frac{6}{2-3x}+\frac{10}{6-5x}+\frac{4}{10+x}=0$.

18. $\frac{1}{x+2}+\frac{1}{x+10}=\frac{1}{x+4}+\frac{1}{x+8}$.

19. $\frac{1}{x-5}+\frac{1}{x+2}=\frac{1}{x-4}+\frac{1}{x+1}$.

20. $\frac{2x+1}{x+1}+\frac{2x+9}{x+5}=\frac{2x+3}{x+2}+\frac{2x+7}{x+4}$.

21. $\frac{3}{b}(x-2a)+\frac{2}{a}(x+b)=1$.

22. $\frac{x+a}{x-b}+\frac{x+b}{x-a}=2$.

23. $\frac{a-b}{bx+c}+\frac{a+b}{ax-c}=0$.

24. $\frac{x+a}{x-a}-\frac{x-b}{x+b}=\frac{2(a+b)}{x}$.

25. $\frac{a^2}{a+x}+\frac{b^2}{b+x}=a+b$.

26. $\frac{x}{a}+\frac{y}{b}=1,\quad \frac{x}{b}-\frac{y}{a}=1$.

27. $\frac{1}{1-x+y}-\frac{1}{x+y-1}=\frac{2}{3},\quad \frac{1}{1-x+y}-\frac{1}{1-x-y}=\frac{4}{3}$.

MISCELLANEOUS EXAMPLES. III.

A.

1. What must be added to $x^3 - 3x^2y + 3xy^2$ that the sum may be

$$3x^2y - 3xy^2 + y^3?$$

2. Shew that

$$x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 4xyz = (y+z)(z+x)(x+y).$$

3. Find the factors of (i) $x^4 - y^4$, (ii) $a^2 - b^2 + 2a - 2b$, and

(iii) $x^2 - 3x - 28$.

4. Find the H.C.F. of $x^4 + 9x - 20$ and $5x^4 + 9x^3 - 64$.

Simplify $$\frac{x^4 + 9x - 20}{5x^5 + 9x^4 - 64x}.$$

5. Solve

(i) $$\frac{x-1}{2} + \frac{2x-7}{3} = x - 2.$$

(ii) $$3x - 4y + 2 = 5x - 6y - 2 = 7x + 2y + 4.$$

(iii) $$\frac{a}{x+b} + \frac{b}{x+a} = \frac{a+b}{x}.$$

6. If *A* were to give 10 shillings to *B*, he would then have three times as much as *B*; but if *B* were to give 5 shillings to *A*, *A* would have four times as much as *B*. How much has each?

B.

1. Find the value of $a(a+b)(a+b+c) - a(a-b)(a-b-c)$, when $a=5$, $b=3$ and $c=-6$; also when $a=-3$, $b=-2$, and $c=4$.

2. Shew that $a^2 + 3(a-2b)^2 = 3(a-b)^2 + (a-3b)^2$, and that

$$\left(a + \frac{1}{a}\right)^2 - \left(b + \frac{1}{b}\right)^2 = \left(ab - \frac{1}{ab}\right)\left(\frac{a}{b} - \frac{b}{a}\right).$$

3. Find the factors of (i) $x^3 - 4x^2 + 4x$, (ii) $2x^2 - 5x + 2$, and

(iii) $x^3 - x^2 + x - 1$.

4. Simplify $\dfrac{1}{x-3} + \dfrac{1}{x+3} - \dfrac{6}{x^2-9}$, and shew that

$$\left(y - \frac{a^2 - xy}{y - x}\right)\left(x + \frac{a^2 - xy}{y - x}\right) + \left(\frac{a^2 - xy}{y - x}\right)^2 = a^2.$$

5. Solve

(i) $\dfrac{1}{x-1}+\dfrac{2}{x-2}=\dfrac{3}{x-3}$,

(ii) $\left.\begin{aligned}\frac{x}{a}-\frac{y}{b}&=1\\ \frac{x}{b}+\frac{y}{a}&=\frac{a}{b}\end{aligned}\right\}$.

6. Find the fraction which becomes $\frac{1}{5}$ when 2 is added to its numerator, and $\frac{1}{6}$ when 2 is taken from its denominator.

C.

1. Prove that

$$(x-1)^2(y^2+1)-(x^2+1)(y-1)^2=2(x-y)(xy-1),$$

and that

$$(x+y)(x+z)-x^2=(y+z)(y+x)-y^2=(z+x)(z+y)-z^2.$$

2. Prove that the product of any two numbers is equal to a quarter of the difference of the square of their sum and the square of their difference.

3. Find the factors of (i) $1+18x-63x^2$, (ii) $3x^3y-24y^4$, and

(iii) $(x^3+3x)^2-(3x^2+1)^2$.

4. Simplify $\dfrac{x^2-10x+21}{x^3-46x-21}$,

and $\dfrac{1}{(x-3)(x-2)}-\dfrac{x-4}{(x-1)(x-3)}+\dfrac{x-3}{(x-1)(x-2)}$.

5. Solve

(i) $\dfrac{x-3}{2}-\dfrac{x-1}{4}+\dfrac{x+1}{6}-\dfrac{x+3}{8}=0$,

(ii) $ax+by=c,\quad a^2x+b^2y=c^2$.

6. *A* and *B* have £100 between them; but if *A* were to lose one-half of his money, and *B* were to lose one-third of his, they would then have only £55 between them. How much has each?

D.

1. Find the value of $\sqrt{a^3+b^3+c^3}-(a-b-c)^2$, when $a=3$, $b=-3$ and $c=4$.

2. Shew that

$$(x-5)(x-4)-3(x-2)(x-1)+3(x+1)(x+2)-(x+4)(x+5)=0,$$

and that

$$1+n+\tfrac{1}{2}n(n+1)+\tfrac{1}{6}n(n+1)(n+2)=\tfrac{1}{6}(n+1)(n+2)(n+3).$$

3. Find the factors of (i) $x^3+x^2y-6xy^2$, (ii) $x^3+ax^2-a^2x-a^3$ and (iii) $x^2y^2-x^2-y^2+1$.

4. Simplify

(i) $$\frac{a-x}{a+x}+\frac{4ax}{a^2-x^2}+\frac{a+x}{a-x}.$$

(ii) $$\frac{x^2+yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y-z)(y-x)}+\frac{z^2+xy}{(z-x)(z-y)}.$$

5. Solve

(i) $$\frac{3x+1}{4}-2(6-x)=\frac{5x-4}{7}-\frac{x-2}{3}.$$

(ii) $$3y+\frac{9}{x}+6=0,\quad y+\frac{5}{x}=8.$$

6. A fox is pursued by a greyhound, and is 60 of her own leaps before him. The fox takes 3 leaps in the time that the greyhound takes 2; but the greyhound goes as far in 3 leaps as the fox does in 7. In how many leaps will the greyhound catch the fox?

E.

1. Shew that

$$(m+n)(m+n-1)=m(m-1)+2mn+n(n-1),$$

and that

$$(m+n)(m+n-1)(m+n-2)=m(m-1)(m-2)$$
$$+3m(m-1)n+3mn(n-1)+n(n-1)(n-2).$$

2. Divide x^4-y^4 by $x-y$, and from the result *write down* the quotient when $(x+y)^4-16z^4$ is divided by $x+y-2z$.

3. Find the factors of

$$5x^2+24x-5, \text{ and of } a^3-2abc-ab^2-ac^2.$$

4. Find the L.C.M. of $8a^3-18ab^2$, $8a^3+8a^2b-6ab^2$ and

$$4a^2-8ab+3b^2.$$

5. Shew that

(i) $$1+\frac{a}{x-a}+\frac{bx}{(x-a)(x-b)}=\frac{x^2}{(x-a)(x-b)},$$

(ii) $$1+\frac{a}{x-a}+\frac{bx}{(x-a)(x-b)}+\frac{cx^2}{(x-a)(x-b)(x-c)}$$
$$=\frac{x^3}{(x-a)(x-b)(x-c)}.$$

6. A debtor is just able to pay his creditors five shillings in the pound; but if his assets had been five times as great, and his debts two-thirds of what they really were, he would have had a balance in his favour of £140. How much did he owe?

F.

1. Find the value of $\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2$ when $a=-1$, $b=-3$ and $c=-5$.

2. Divide $x^8-2a^4x^4+a^8$ by $x^3+ax^2+a^2x+a^3$.

3. Find the L.C.M. of x^2-1, x^2+1, $(x-1)^2$, $(x+1)^2$, x^3-1 and x^3+1.

4. Add together $\frac{1}{2x+y}$, $\frac{1}{2x-y}$ and $\frac{3x}{y^2-4x^2}$; and shew that

$$\frac{(x-a)(y-a)}{(a-b)(a-c)}+\frac{(x-b)(y-b)}{(b-c)(b-a)}+\frac{(x-c)(y-c)}{(c-a)(c-b)}=1.$$

5. Solve

(i) $$\frac{1}{x+a}+\frac{1}{x+b}=\frac{2}{x+a+b},$$

(ii) $$\left.\begin{aligned} 5x+2y+3z&=13\\ 3x+7y-\ z&=\ 2\\ x-2y+\ z&=\ 5\end{aligned}\right\}.$$

6. When the arable land of a farm was let at 30 shillings, and the pasture at 40 shillings an acre, the total rent of a farm was £550. When the rent of the pasture was reduced by 5 shillings an acre, and that of the arable land by 10 shillings an acre, the whole rent was £387. 10*s*. What was the total acreage of the farm?

CHAPTER XIV.

QUADRATIC EQUATIONS.

134. IT is clear that a product is zero when any one of its factors is zero; and it is also clear that a product can never be zero unless one of its factors is zero. Thus ab is zero if a is zero, or if b is zero; and, if we know that ab is zero, we conclude that *either* a or b must be zero.

So also abc is zero if a, or b, or c is zero; and, if we know that abc is zero, we conclude that either a, or b, or c must be zero; and so on, however many factors there may be.

Similarly the product $(x-2)(x-4)$ is zero, if $x-2$ is zero, or if $x-4$ is zero; and, in order that the product may be zero, one of these factors must be zero.

Thus the equation

$$(x-2)(x-4)=0$$

is satisfied if $x-2=0$, or if $x-4=0$, that is, if $x=2$, or if $x=4$, and in no other case. The *roots* of the equation are therefore 2 and 4.

Again, the equation

$$(x-3)(x-4)(x-5)=0$$

is satisfied if $x-3=0$, or if $x-4=0$, or if $x-5=0$, and in no other case; so that in order that the equation may be true x must be either 3, 4 or 5: thus 3, 4 and 5 are the *roots* of the equation

$$(x-3)(x-4)(x-5)=0.$$

Thus the equation

$$(x-a)(x-b)(x-c)\ldots\ldots=0$$

is equivalent to the system of *alternative* equations

$$x-a=0, \ \textit{or} \ x-b=0, \ \textit{or} \ x-c=0, \ \&c.$$

135. From the examples considered in the last article it will be apparent that the solution of an equation of any degree can be written down at once provided the equation is given in the form of a product of factors of the first degree equated to zero.

The following are examples of such equations:

Ex. 1. Solve $(x-1)(x+1)=0$.

The equation is satisfied if $x-1=0$, or if $x+1=0$, and in no other case.

Hence we must have

$$x-1=0, \text{ or } x+1=0,$$

that is

$$x=1, \text{ or } x=-1.$$

Thus the roots of the equation are 1 and -1.

Ex. 2. Solve $x(x+1)(x+2)=0$.

The equation is satisfied if $x=0$, or if $x+1=0$, or if $x+2=0$, and in no other case.

Hence we must have

$$x=0, \text{ or } x+1=0, \text{ or } x+2=0,$$

that is

$$x=0, \text{ or } x=-1, \text{ or } x=-2.$$

Thus the roots of the equation are 0, -1 and -2.

Ex. 3. Solve $x(2x-1)(2x+3)=0$.

The equation is satisfied if $x=0$, or if $2x-1=0$, or if $2x+3=0$, and in no other case.

Hence we must have

$$x=0, \text{ or } 2x-1=0, \text{ or } 2x+3=0,$$

that is $$x=0, \text{ or } x=\tfrac{1}{2}, \text{ or } x=-\tfrac{3}{2}.$$

Thus the roots of the equation are $0, \frac{1}{2}, -\frac{3}{2}$.

136. Since all the terms of any equation can be transposed to one side, an equation can always be written with all its terms on one side of the sign of equality, and zero on the other side.

It therefore follows from the last article that *the problem of solving an equation of any degree is the same as the problem of finding the factors of an expression of the same degree.*

Hence the process of solving *any* equation which involves only one unknown quantity is as follows:—First write the equation with all its terms on one side of the sign of equality, and zero on the other side; then resolve the whole expression into factors, and the values obtained by equating each of these factors separately to zero will be the required roots.

In the following examples the resolution into factors can be performed by inspection.

Ex. 1. Solve the equation $x^2-3x=0$.

Since $$x^2-3x=x(x-3),$$

we have $$x(x-3)=0.$$

Whence $$x=0, \text{ or } x-3=0;$$

the roots required are therefore 0 and 3.

Ex. 2. Solve the equation $x^2-9=0$.

Since $\qquad x^2-9=(x-3)(x+3)$,

we have $\qquad (x-3)(x+3)=0$.

Whence $\qquad x-3=0$, or $x+3=0$;

the roots required are therefore 3 and -3.

Ex. 3. Solve the equation $x^2-2=0$.

Since $\qquad x^2-2=(x-\sqrt{2})(x+\sqrt{2})$,

we have $\qquad (x-\sqrt{2})(x+\sqrt{2})=0$.

Hence $\qquad x-\sqrt{2}=0$, or $x+\sqrt{2}=0$.

Thus $\sqrt{2}$ and $-\sqrt{2}$ are the required roots of the equation.

Ex. 4. Solve the equation $x^3-4x=0$.

Since $\qquad x^3-4x=x(x^2-4)=x(x-2)(x+2)$,

we have $\qquad x(x-2)(x+2)=0$.

Hence $\qquad x=0$, or $x-2=0$, or $x+2=0$.

Thus 0, 2 and -2 are the roots of the equation.

Ex. 5. Solve $9x^3=4x$.

Transposing, we have

$$9x^3-4x=0,$$

that is $\qquad x(9x^2-4)=0$,

or $\qquad x(3x-2)(3x+2)=0$.

Hence $\qquad x=0$, or $3x-2=0$, or $3x+2=0$,

that is $\qquad x=0$, or $x=\frac{2}{3}$, or $x=-\frac{2}{3}$.

Ex. 6. Solve $x^2+6=5x$.

Transposing, we have

$$x^2-5x+6=0,$$

that is $$(x-2)(x-3)=0.$$

Hence $$x-2=0, \text{ or } x-3=0.$$

Thus 2 and 3 are the required roots.

EXAMPLES. XXXV.

Solve the following equations:—

1. $(3x-1)(3x+1)=0$.
2. $(x+1)(x+2)=0$.
3. $x(2x-1)(2x+1)=0$.
4. $(x-2)(x-1)(x+1)(x+2)=0$.
5. $3x^2=\frac{1}{2}(x^2+5)$.
6. $5(x^2+9)=3(x^2+25)$.
7. $5(x^2+4)=4(x^2+9)$.
8. $2(x^2+7)=7(x^2+2)$.
9. $x^2+7x=0$.
10. $x^2-11x=0$.
11. $3x^2=4x$.
12. $5x^2=6x$.
13. $ax^2-bx=0$.
14. $cx^2+c^2x=0$.
15. $5(x^2+3)-(x-5)(x+5)=76$.
16. $7(x^2-1)-(x+3)(x-3)-56=0$.
17. $3x^2+(5x+2)^2=20x+32$.
18. $17+3x=\frac{1}{2}(x+3)^2-28$.
19. $x^2-3x+2=0$.
20. $x^2-7x+10=0$.
21. $x^2-2x=8$.
22. $x^2+x=6$.
23. $x^3-2x^2-8x=0$.
24. $x^3+x^2=6x$.
25. $(x+a)(x+b)(x+c)=0$.
26. $(x+b+c)(x+c+a)(x+a+b)=0$.

137. A *quadratic equation* is an equation which contains the unknown quantity to the second, but to no higher power. Thus $x^2=4$, $3x^2+4x=7$, and $ax^2+bx+c=0$ are quadratic equations.

When all the terms are transposed to one side, a quadratic equation must be of the form

$$ax^2 + bx + c = 0,$$

where a, b, c are supposed to represent known quantities.

We proceed to shew how to solve any quadratic equation.

As we have already remarked, the whole difficulty consists in finding the factors of the expression on the left side of the equation; and we have shewn in Art. 100 how this can be done by changing the expression into an equivalent one which is the difference of two squares.

138. To solve the equation

$$ax^2 + bx + c = 0.$$

Divide by a, the coefficient of x^2; the equation then becomes

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Now add and subtract *the square of half the coefficient of* x, that is the square of $\frac{1}{2}\frac{b}{a}$. Then we have

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0.$$

The first three terms are now a perfect square, namely

$$\left(x + \frac{b}{2a}\right)^2.$$

Hence we have $$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0,$$

that is $$\left(x + \frac{b}{2a}\right)^2 - \left\{\sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}\right\}^2 = 0.$$

Hence [Art. 96]

$$\left\{x+\frac{b}{2a}-\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}\left\{x+\frac{b}{2a}+\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}=0.$$

We therefore have

$$x+\frac{b}{2a}-\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}=0,$$

or else $$x+\frac{b}{2a}+\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}=0.$$

That is $$x=-\frac{b}{2a}+\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)},$$

or $$x=-\frac{b}{2a}-\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}.$$

Thus there are *two* roots of a quadratic equation.

Ex. 1. Solve $x^2+12x+35=0$.

Add and subtract the square of half the coefficient of x, that is the square of 6. The equation then becomes

$$x^2+12x+6^2-6^2+35=0,$$

that is $$(x+6)^2-1=0.$$

$$\therefore\ \{(x+6)+1\}\{(x+6)-1\}=0,$$

that is $$(x+7)(x+5)=0.$$

Hence $$x+7=0,\text{ or }x+5=0.$$

Thus $$x=-7,\text{ or }x=-5.$$

Ex. 2. Solve $3x^2=10x-3$.

By transposition we have

$$3x^2-10x+3=0.$$

Divide by 3, the coefficient of x^2; the equation then becomes

$$x^2 - \tfrac{10}{3}x + 1 = 0,$$

Half the coefficient of x is $-\tfrac{5}{3}$.

Adding and subtracting $(-\tfrac{5}{3})^2$, that is $\tfrac{25}{9}$, the equation becomes

$$x^2 - \tfrac{10}{3}x + \tfrac{25}{9} - \tfrac{25}{9} + 1 = 0,$$

or

$$(x - \tfrac{5}{3})^2 - \tfrac{16}{9} = 0,$$

$$\therefore\ \{x - \tfrac{5}{3} + \tfrac{4}{3}\}\{x - \tfrac{5}{3} - \tfrac{4}{3}\} = 0;$$

$$\therefore\ (x - \tfrac{1}{3})(x - 3) = 0.$$

Hence

$$x - \tfrac{1}{3} = 0, \text{ or } x - 3 = 0.$$

Thus

$$x = \tfrac{1}{3}, \text{ or } x = 3.$$

NOTE. In most cases the factors can be written down at once, as in Art. 98, without completing the square. The student should always see if this can be done: he will thus save himself a great deal of unnecessary work.

Ex. 3. Solve $x^2 = 4x + 9$.

By transposition, we have

$$x^2 - 4x - 9 = 0.$$

Add and subtract 4, the square of half the coefficient of x. Then

$$x^2 - 4x + 4 - 4 - 9 = 0,$$

that is

$$(x - 2)^2 - 13 = 0;$$

$$\therefore\ (x - 2 + \sqrt{13})(x - 2 - \sqrt{13}) = 0.$$

Hence

$$x - 2 + \sqrt{13} = 0, \text{ or } x - 2 - \sqrt{13} = 0.$$

Thus

$$x = 2 - \sqrt{13}, \text{ or } x = 2 + \sqrt{13}.$$

139. Instead of working out every example from the beginning, we may use the general formulae found in Art. 138, and substitute for a, b and c their values in the particular equation we are considering. Thus, the roots of the equation $ax^2 + bx + c = 0$ being $-\frac{b}{2a} \pm \sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}$, we find the roots of $3x^2 - 10x + 3 = 0$ by putting 3 for a, -10 for b and 3 for c in the above formulae.

Hence the roots of $3x^2 - 10x + 3 = 0$ are

$$-\frac{-10}{6} \pm \sqrt{\left(\tfrac{100}{36} - \tfrac{3}{3}\right)},$$

that is $\frac{10}{6} \pm \frac{8}{6}$; and hence the roots are 3 and $\frac{1}{3}$.

140. In Art. 138 we found that the quadratic equation $ax^2 + bx + c = 0$ had *two* roots, namely

$$-\frac{b}{2a} + \sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)} \text{ and } -\frac{b}{2a} - \sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}.$$

If $\frac{b^2}{4a^2} - \frac{c}{a} = 0$, both the roots reduce to $-\frac{b}{2a}$, and are thus equal to one another. In this case we do not say that the equation has only *one* root, but that it has *two equal* roots.

It is clear that the roots are unequal unless $\frac{b^2}{4a^2} - \frac{c}{a}$ is zero.

Hence in order that the roots of $ax^2 + bx + c = 0$ may be equal to one another, it is necessary and sufficient that $\frac{b^2}{4a^2} - \frac{c}{a} = 0$, or that $b^2 - 4ac = 0$.

When $b^2 - 4ac = 0$, the expression $ax^2 + bx + c$ is a perfect square *so far as x is concerned;* for $ax^2 + bx + c$, that is $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$, will then be $a\left(x+\frac{b}{2a}\right)^2$, since $\frac{b^2}{4a^2}=\frac{c}{a}$.

141. If $\frac{b^2}{4a^2}-\frac{c}{a}$ is negative, $\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}$ is *imaginary* [Art. 102].

Hence the roots of $ax^2+bx+c=0$ are imaginary if $\frac{b^2}{4a^2}-\frac{c}{a}$ is negative, or if b^2-4ac is negative.

It should be noticed that if either of the roots of a quadratic equation is imaginary, they are *both* imaginary.

142. By multiplying both sides of an equation by the same quantity, we do not destroy the equality, and therefore the new equation is satisfied by all the values which satisfy the original equation; if, however, we multiply by any integral expression *which contains the unknown quantity*, the new equation will be satisfied not only by the values which satisfy the original equation, but *also* by any value which makes the expression by which we multiply vanish.

For example, the equation $x^2=9$ is satisfied by the values $x=3$, or $x=-3$. If we multiply both sides of the equation by $x-2$, we have the new equation

$$x^2(x-2)=9(x-2);$$

and this new equation is satisfied not only by $x=3$, and by $x=-3$, but also by $x=2$.

Thus *additional roots* are introduced whenever both sides of an equation are *multiplied* by any integral expression which contains the unknown quantity.

When an equation contains fractions in whose denominators the unknown quantity occurs, the equation may be multiplied by the L. C. M. of the denominators without introducing any additional roots; for we cannot divide both sides of the resulting equation by any one of the factors of the L. C. M. without re-introducing fractions.

The student must, however, be careful to multiply by the *lowest* common multiple of the denominators, for otherwise the resulting equations will have roots which are not roots of the original equation.

We may, for example, multiply both sides of the equation

$$\frac{2x}{x-1}-\frac{10}{x^2-1}=\frac{7}{x+1}$$

by x^2-1, the L. C. M. of the denominators of the fractions: we thus obtain

$$2x(x+1)-10=7(x-1),$$

and this new equation is not satisfied by either of the values obtained by equating to zero the factor by which we have multiplied, and hence no additional roots have been introduced by the multiplication.

If, however, we multiply the original equation by $(x-1)(x^2-1)$, we shall have

$$2x(x+1)(x-1)-10(x-1)=7(x-1)^2,$$

and one of the roots of this equation, namely $x=1$, is not a root of the original equation.

The student should always remember that when both sides of an equation are divided by any factor *which contains the unknown quantity*, the resulting equation will not give *all* the roots of the original equation; to find the remaining roots we must equate to zero the factor by which we have divided.

For example, if we divide both sides of the equation

$$(x^2-4)(x+2) = (x^2-4)(2x+1)$$

by x^2-4, we obtain the equation

$$x+2 = 2x-1,$$

from which we obtain $x=3$. But $x=3$ is not the *only* solution of the original equation; to obtain the other solutions we must equate x^2-4 to zero.

EXAMPLES. XXXVI.

Solve the following equations:—

1. $(x+1)^3=(x-1)^3+26.$
2. $(x-1)^3=(x+1)^3-56.$
3. $(x-3)^2=x+3.$
4. $(x-4)^2=x-2.$
5. $(x-1)(x-2)=20.$
6. $(x+1)(x+3)=2(x+2).$
7. $4x+3=(x+2)x.$
8. $4x-3=(2-x)x.$
9. $9x^2+6x+1=0.$
10. $9x^2+16=24x.$
11. $7x^2-50x+7=0.$
12. $3x^2-11x-4=0.$
13. $x^2+21x+110=0.$
14. $110x^2-21x+1=0.$
15. $21x^2-13x-20=0.$
16. $21x^2+23x-20=0.$
17. $6x^2+13x+6=0.$
18. $6x^2-5x=1.$
19. $29x^2-41x-138=0.$
20. $29x^2+11x-138=0.$
21. $9x^2-63x+68=0.$
22. $16x^2+3=16x.$
23. $x^2-16=215-10x.$
24. $(x+2)^2=4(x-1)^2.$
25. $x+\frac{1}{x}=\frac{5}{2}.$
26. $x+\frac{1}{x}=6.$

27. $\frac{x}{x-1}+\frac{x-1}{x}=\frac{13}{6}.$

28. $\frac{x+1}{x+2}+\frac{x+2}{x+1}=\frac{29}{10}.$

29. $\frac{1}{x-2}-\frac{2}{x+2}=\frac{3}{5}.$

30. $\frac{7}{x+4}+\frac{1}{4-x}=\frac{5}{3}.$

31. $\frac{x}{x-2}+\frac{4}{x+1}=3.$

32. $\frac{5}{x+1}-\frac{10}{x+10}=\frac{2}{3x-3}.$

33. $\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{5x}{x^2-1}.$

34. $\frac{1}{2}+\frac{1}{3+x}+\frac{1}{2+3x}=0.$

35. $\frac{1}{x}+\frac{1}{x+4}=\frac{1}{x+1}+\frac{1}{x+2}.$

36. $\frac{x}{x+1}+\frac{x+1}{x+2}=\frac{x-2}{x-1}+\frac{x-1}{x}.$

37. $3\,\frac{x-1}{x+1}-2\,\frac{x+1}{x-1}=5.$

38. $\frac{3}{x-1}+\frac{4}{x-3}=\frac{15}{x+3}.$

39. $\frac{2x-1}{2x+1}+\frac{13}{11}=\frac{3x+5}{3x-5}.$

40. $\frac{x}{x+1}+\frac{x+1}{x+2}+\frac{x+2}{x+3}=3.$

41. $x^2+2a^2=3ax.$

42. $x^2+3a^2=4ax.$

43. $4x^2+4ax=b^2-a^2.$

44. $9x^2-6ax=b^2-a^2.$

45. $x^2+2ab=2ax+b^2.$

46. $x^2+2ax=2ab+b^2.$

47. $x^2-2(a-b)\,x+b^2=2ab.$

48. $x^2+2(b-c)\,x+c^2=2bc.$

49. $(b-c)\,x^2+(c-a)\,x+(a-b)=0.$

50. $(a+b)\,x^2+cx-a-b-c=0.$

51. $abx^2-(a^2+b^2)\,x+ab=0.$

52. $bcx^2+(b^2+c^2)\,x+bc=0.$

53. $(a^2-b^2)\,(x^2-1)=4abx.$

54. $(b^2-a^2)\,(x^2+1)=2\,(a^2+b^2)\,x.$

55. $x+\frac{1}{x}=a+\frac{1}{a}.$

56. $\frac{a}{a+x}+\frac{a}{a-x}=4.$

57. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{a}+\frac{1}{b}.$

58. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a}{b}+\frac{b}{a}.$

59. $\frac{a^2}{x-b}+\frac{b^2}{x-a}=a+b.$

60. $\frac{x}{a}+\frac{a}{x}=\frac{b}{a}+\frac{a}{b}.$

61. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{c+a}+\frac{1}{c+b}.$

62. $\frac{x}{x+a}+\frac{x}{x+b}=\frac{c}{c+a}+\frac{c}{c+b}.$

63. $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}.$

143. We have seen that if

$$x + px + q = (x - \alpha)(x - \beta),$$

then α and β are the roots of the equation

$$x^2 + px + q = 0.$$

The converse proposition, namely that if α and β are roots of the equation

$$x^2 + px + q = 0,$$

then $x - \alpha$ and $x - \beta$ are factors of the expression

$$x^2 + px + q,$$

may be proved independently as follows.

Since α is a root of the quadratic equation

$$x^2 + px + q = 0,$$

we have $$\alpha^2 + p\alpha + q = 0.$$

Hence $$\begin{aligned} x^2 + px + q &= x^2 + px + q - (\alpha^2 + p\alpha + q) \\ &= x^2 - \alpha^2 + px - p\alpha \\ &= (x - \alpha)(x + \alpha + p); \end{aligned}$$

$\therefore$ $x - \alpha$ is a factor of $x^2 + px + q$.

Hence, if α be a root of the *equation* $x^2 + px + q = 0$, $x - \alpha$ will be a factor of the *expression* $x^2 + px + q$.

If β be also a root of the equation $x^2 + px + q = 0$, then $x - \beta$ will similarly be a factor of the expression $x^2 + px + q$.

Now the expression $x^2 + px + q$ cannot have more than *two* factors of the form $x - \alpha$, for the product of any number of factors of the form $x - \alpha$ must be of the same degree in x as the number of the factors; also $x^2 + px + q$ clearly has no factors not containing x.

Hence $$x^2 + px + q = (x - \alpha)(x - \beta).$$

144. We have proved in the last Article that if α and β are the roots of the equation $x^2+px+q=0$, then will

$$x^2+px+q=(x-\alpha)(x-\beta),$$

that is $$x^2+px+q=x^2-(\alpha+\beta)x+\alpha\beta.$$

Hence we must have

and $$\left.\begin{aligned}\alpha+\beta&=-p\\ \alpha\beta&=q\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Thus in the equation

$$x^2+px+q=0,$$

the *sum of the roots* $=-p$,

and the *product of the roots* $=q$.

The equation

$$ax^2+bx+c=0$$

becomes, on dividing by a,

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Hence, from (i), if α, β be the roots of $x^2+\frac{b}{a}x+\frac{c}{a}=0$, that is of $ax^2+bx+c=0$, we have

and $$\left.\begin{aligned}\alpha+\beta&=-\frac{b}{a}\\ \alpha\beta&=\frac{c}{a}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Thus in the equation

$$ax^2+bx+c=0,$$

the *sum of the roots* $=-\frac{b}{a}$,

and the *product of the roots* $=\frac{c}{a}$.

The above relations between the roots of a quadratic equation and the coefficients of the different powers of the unknown quantity, are of great importance.

Analogous relations hold good for equations of the third and of higher degrees.

145. Although we cannot in all cases find the roots of a *given equation*, it is very easy to solve the converse problem, namely the problem of finding an equation which has *given roots*.

For example, to find the equation whose roots are 4 and 5.

We want to find an equation which is satisfied when $x=4$, or when $x=5$; that is when $x-4=0$, or when $x-5=0$; and in no other case. The equation required must be

$$(x-4)(x-5)=0,$$

for this is an equation which is a true statement when $x-4=0$, or when $x-5=0$, and in no other case.

If we get rid of the brackets by multiplying out, the equation becomes $x^2-9x+20=0$*.

Again, to find the equation whose roots are 2, 3 and -4.

We have to find an equation which is satisfied when $x-2=0$, or when $x-3=0$, or when $x+4=0$, and in no other case. The equation must therefore be

$$(x-2)(x-3)(x+4)=0,$$

that is

$$x^3-x^2-14x+24=0.$$

---

* The equation $x^2-9x+20=0$ is certainly *an* equation with the proposed roots; but to prove that it is the *only* equation with the proposed and with no other roots we must assume that *every equation has a root.*

If the equation $x^5+7x^2-2=0$, for example, had no roots, then $(x-4)(x-5)(x^5+7x^2-2)=0$ would be an equation with the proposed roots and with no others.

The proof of the proposition that every equation has a root is much too difficult for an elementary book.

Similarly, the equation whose roots are 0, -1, and $-\frac{1}{2}$ is

$$x(x+1)(x+\tfrac{1}{2})=0,$$

that is
$$x^3+\tfrac{3}{2}x^2+\tfrac{1}{2}x=0.$$

146. We will conclude this chapter by the solution of some examples.

Ex. 1. Shew that no numerical value of x will make x^2-4x+5 zero; and find its least value.

Since
$$x^2-4x+5=(x-2)^2+1,$$
and $(x-2)^2$ is always positive, $1+(x-2)^2$ is always greater than 1, except when $x-2=0$, and then it is equal to 1.

Thus x^2-4x+5 can never be zero, and its least value is 1.

Ex. 2. Find the equation whose roots are the squares of the roots of the equation $x^2+5x-7=0$.

Let α, β be the roots of the given equation; then α^2, β^2 will be the roots of the required equation. Hence the required equation is

$$(x-\alpha^2)(x-\beta^2)=0,$$

that is
$$x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0 \quad \text{.................(i).}$$

We have therefore to find $\alpha^2+\beta^2$ and $\alpha^2\beta^2$. Now, by Art. 144, we have

$$\alpha+\beta=-5,$$

and
$$\alpha\beta=-7.$$

Hence
$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=25+14=39;$$

also
$$\alpha^2\beta^2=49.$$

Substituting in (i), we have for the required equation

$$x^2-39x+49=0.$$

We might have obtained the required equation by finding the roots of the given equation, and squaring those roots; but it is best to use the relations found in Art. 144.

Ex. 3. If α, β are the roots of the equation

$$ax^2 + bx + c = 0,$$

find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

The required equation is

$$\left(x - \frac{\alpha}{\beta}\right)\left(x - \frac{\beta}{\alpha}\right) = 0,$$

that is $$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + 1 = 0.$$

Now $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta};$$

and, by Art. 144, we have

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

Hence $$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \frac{b^2}{a^2} - 2\frac{c}{a};$$

$$\therefore \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c}{a}} = \frac{b^2}{ac} - 2.$$

Hence the required equation is

$$x^2 - \left(\frac{b^2}{ac} - 2\right)x + 1 = 0,$$

or $$acx^2 - (b^2 - 2ac)\,x + ac = 0.$$

Ex. 4. Solve the equation

$$\sqrt{(x^2-9)}+x=9.$$

If we write the equation with the radical on one side, and all the other terms on the other side of the sign of equality, and then square both sides, we shall obtain an equation free from radicals. Thus

$$\sqrt{(x^2-9)}=9-x.$$

Square both sides; then

$$x^2-9=81-18x+x^2;$$

$$\therefore\ 18x=90;$$

$$\therefore\ x=5.$$

Ex. 5. Solve the equation

$$\sqrt{2x+8}-2\sqrt{x+5}=2.$$

Squaring both sides of the equation, we have

$$2x+8+4(x+5)-4\sqrt{2x+8}\sqrt{x+5}=4,$$

or

$$6x+24-4\sqrt{2x+8}\sqrt{x+5}=0;$$

$$\therefore\ 3x+12=2\sqrt{2x+8}\sqrt{x+5}.$$

Square both sides of this last equation, and we have

$$9x^2+72x+144=4(2x^2+18x+40);$$

$$\therefore\ x^2-16=0.$$

Hence $x=4$, or $x=-4$.

Now if we put 4 for x in the original equation, we should have $\sqrt{16}-2\sqrt{9}=2$,

or $4-6=2$,

which is not true.

And if we put -4 for x, we should have

$$\sqrt{0}-2\sqrt{1}=2,$$

which is not true.

So that neither of the values we have found for x appears to satisfy the equation.

The failure is however only apparent; and is due to our having supposed that $\sqrt{2x+8}$ and $\sqrt{x+5}$ were necessarily positive, whereas every algebraical quantity has *two* square roots, one positive and the other negative.

Bearing this in mind we shall find that $x=4$ does satisfy the given equation, for the condition is

$$\pm\sqrt{16}-2(\pm\sqrt{9})=2,$$

or

$$\pm 4-(\pm 6)=2,$$

which is true when the lower signs are taken.

Similarly we can see that $x=-4$ satisfies the given equation.

Since we cannot fix the sign of the square root of an algebraical quantity, the four equations

$$\sqrt{2x+8}-2\sqrt{x+5}=2,$$

$$\sqrt{2x+8}+2\sqrt{x+5}=2,$$

$$-\sqrt{2x+8}-2\sqrt{x+5}=2,$$

and

$$-\sqrt{2x+8}+2\sqrt{x+5}=2,$$

although different in appearance, are really the *same equation.*

EXAMPLES XXXVII.

1. Write down the equations whose roots are

(i) 2 and -2, (ii) 0, 3, and -4, (iii) $-\frac{1}{2}$ and $-\frac{1}{3}$,
(iv) a, $-b$ and c, (v) $2-\sqrt{2}$ and $2+\sqrt{2}$,
and (vi) $a+\sqrt{b}$ and $a-\sqrt{b}$.

2. Write down the product of the roots of each of the following equations:

(i) $x^2+4x+1=0$, (ii) $x^2+7x-2=0$, and (iii) $3x^2+5x-7=0$.

3. Write down the sum of the roots of each of the following equations:

(i) $x^2-4a^2=0$, (ii) $x^2+3x-5=0$, and (iii) $5x^2+7x-8=0$.

4. Find the sum of the squares of the roots of the equation

$$x^2+4x+2=0.$$

5. Find the sum of the squares of the roots of the equation

$$x^2+px+q=0.$$

6. Shew that the roots of the equation $x^2-186x+25=0$ are the squares of the roots of the equation $x^2-14x+5=0$.

7. Shew that the sum of the squares of the roots of the equation $x^2+ax+b=0$ is the same as the sum of the squares of the roots of the equation $x^2+3ax+b+4a^2=0$.

8. For what value of a are the roots of the equation

$$3x^2+4x+a=0$$

equal to one another?

9. For what values of a are the roots of the equation

$$4x^2+(1+a)x+1=0$$

equal to one another?

10. Shew that one of the roots of $100x^2+60x+8=0$ is double the other.

11. Shew that one of the roots of the equation $x^2+px+q=0$ is double the other, if $9q=2p^2$.

12. If α, β are the roots of $2x^2-5x+3=0$, shew that the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is $6x^2-13x+6=0$.

13. If α, β are the roots of $ax^2+bx+c=0$, then will

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{b}{c}=0.$$

14. Shew that, if α, β are the roots of $px^2+qx+r=0$, the equation whose roots are $\alpha+\beta$ and $\frac{\alpha\beta}{\alpha+\beta}$ is $pqx^2+(pr+q^2)x+qr=0$.

15. Find the condition that the difference of the roots of

$$ax^2 + bx + c = 0$$

may be the same as the difference of the roots of

$$px^2 + qx + r = 0.$$

Solve the following equations:

16. $\sqrt{x+10} + \sqrt{x+1} = 1.$

17. $x - 2 + 3\sqrt{x-2} = 0.$

18. $\sqrt{9+4x} = 2x - 3.$

19. $3x = 5 + \sqrt{30x - 71}.$

20. $2x - 5\sqrt{x} = 3.$

21. $x + 3 + \sqrt{x+3} = 6.$

22. $2x + 1 = \sqrt{6x+3}.$

23. $\sqrt{x} - \sqrt{4-x} = \sqrt{4-3x}.$

24. $\sqrt{7x+1} - \sqrt{3x+10} = 1.$

25. $\sqrt{2x+7} + \sqrt{3x-18} = \sqrt{7x+1}.$

CHAPTER XV.

EQUATIONS OF HIGHER DEGREE THAN THE SECOND.

147. It is beyond the range of this book to shew how to solve equations of higher degree than the second, when the equations are in their most general forms; we give however some easy examples of such equations.

Ex. 1. Solve $x^4 - 6x^2 + 8 = 0.$

Since $x^4 - 6x^2 + 8 = (x^2 - 4)(x^2 - 2),$

we have $(x^2 - 4)(x^2 - 2) = 0;$

$$\therefore \quad x^2 - 4 = 0, \text{ giving } x = \pm 2,$$

or $x^2 - 2 = 0$, giving $x = \pm\sqrt{2}.$

Thus there are four roots,

$$\pm 2, \quad \pm\sqrt{2}.$$

Ex. 2. Solve $(x^2 + x)^2 + 4(x^2 + x) - 12 = 0.$

In this, as in the former example, the unknown quantity occurs in two terms, one of which is the *square* of the other. In all such cases we can bring all the terms over to the left side of the sign of equality, and then resolve the expression on the left into two factors just as in the preceding chapter.

In the present instance the factors can be seen by inspection.

For $$A^2 + 4A - 12 = (A + 6)(A - 2);$$
therefore the equation may be written
$$\{(x^2 + x) + 6\}\{(x^2 + x) - 2\} = 0.$$

Hence $$x^2 + x + 6 = 0, \quad \text{or} \quad x^2 + x - 2 = 0.$$

The roots of
$$x^2 + x + 6 = 0 \quad \text{are} \quad -\tfrac{1}{2} \pm \frac{\sqrt{-23}}{2}.$$

The roots of
$$x^2 + x - 2 = 0 \quad \text{are } 1 \text{ and } -2.$$

Hence the given equation has the four roots
$$1, \quad -2, \quad -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-23}.$$

Ex. 3. Solve
$$\frac{x^2}{x+2} + \frac{x+2}{x^2} = \frac{37}{6}.$$

Here one of the terms in which x occurs is the reciprocal of the other. If we put $y = \dfrac{x^2}{x+2}$, we find y from the quadratic equation
$$y + \frac{1}{y} = \frac{37}{6},$$
or, multiplying by $6y$,
$$6y^2 - 37y + 6 = 0,$$
that is $$(6y - 1)(y - 6) = 0.$$

Hence $$y = \tfrac{1}{6}, \quad \text{or} \quad y = 6.$$

Thus $$\frac{x^2}{x+2} = \frac{1}{6}, \quad \text{or} \quad \frac{x^2}{x+2} = 6.$$

In the first case, we have $6x^2 - x - 2 = 0$, the roots of which are $\frac{2}{3}$ and $-\frac{1}{2}$.

In the second case, we have
$$x^2 - 6x - 12 = 0,$$
the roots of which are $3 \pm \sqrt{21}$.

Hence the given equation has the four roots,

$$\tfrac{2}{3},\quad -\tfrac{1}{2},\quad 3\pm\sqrt{21}.$$

Ex. 4. Solve

$$4x^2-6x+3\sqrt{(2x^2-3x+7)}=30.$$

Put $\sqrt{(2x^2-3x+7)}=y,$

then $2x^2-3x+7=y^2,$

$$2x^2-3x=y^2-7,$$

and $4x^2-6x=2y^2-14.$

Hence, from the given equation, we have

$$2y^2-14+3y=30,$$

or $2y^2+3y-44=0,$

that is $(y-4)(2y+11)=0;$

hence $y=4,$ or $y=-\tfrac{11}{2};$

$$\therefore\quad y^2=16,\quad\text{or}\quad y^2=\tfrac{121}{4}.$$

Since $y^2=2x^2-3x+7,$ we have

I. $2x^2-3x+7=16,$

or $2x^2-3x-9=0,$

that is $(2x+3)(x-3)=0;$

$$\therefore\quad x=3,\quad\text{or}\quad x=-\tfrac{3}{2}.$$

II. $2x^2-3x+7=\tfrac{121}{4},$

or $2x^2-3x-\tfrac{93}{4}=0,$

the roots of which are $\dfrac{3}{4}\pm\dfrac{\sqrt{195}}{4}.$

148. We have proved in Art. 143 that if x^2+px+q vanishes when any particular value a is given to x, then $x-a$ is a factor of x^2+px+q.

The same proposition holds good for all rational and integral expressions; but we defer the general proof to a subsequent chapter.

For example, x^3-7x+6 vanishes when 2 is put for x; therefore by the above theorem $x-2$ is a factor, and by division we find that

$$x^3-7x+6=(x-2)(x^2+2x-3).$$

Again, x^3-4x^2+2x+1 vanishes when $x=1$; therefore $x-1$ is a factor, and by division we find that

$$x^3-4x^2+2x+1=(x-1)(x^2-3x-1).$$

149. It follows from the theorem enunciated in the last Article that when one root of an equation is known, the degree of the equation can be lowered.

Ex. 1. One root of the equation $x^3-7x+6=0$ is 2; find the other roots.

Since x^3-7x+6 vanishes when $x=2$, $x-2$ must be a factor, and by division we find that

$$x^3-7x+6=(x-2)(x^2+2x-3).$$

Hence $$(x-2)(x^2+2x-3)=0.$$

Hence the other roots of the equation are those given

by $$x^2+2x-3=0,$$

that is $$(x+3)(x-1)=0.$$

Thus the cubic equation $x^3-7x+6=0$ has the three roots 1, 2, and -3.

Ex. 2. Solve the equation $x^3 - 1 = 0$.

Since $x^3 - 1 = (x-1)(x^2 + x + 1)$,

we have $(x-1)(x^2 + x + 1) = 0$.

Hence $x = 1$;

or else $x^2 + x + 1 = 0$,

the roots of which are

$$-\tfrac{1}{2} \pm \sqrt{(-\tfrac{3}{4})}.$$

Hence there are three roots of the equation $x^3 = 1$, one being real and the other two imaginary. Thus there are three quantities whose cubes are equal to 1; that is, there are *three cube roots* of 1, which are

$$1,\ -\tfrac{1}{2} + \sqrt{(-\tfrac{3}{4})} \text{ and } -\tfrac{1}{2} - \sqrt{(-\tfrac{3}{4})}.$$

Ex. 3. For what values of x will

$$x^4 - 25x^2 + 30x - 9,$$

and $x^4 - 8x^3 + 19x^2 - 14x + 3$

both vanish?

If both expressions vanish for the same value of x, say $x = a$, they will both have $x - a$ as a factor. Now we can find the common factor of any two expressions by the ordinary process of finding their H.C.F.

In the present case the H.C.F. is

$$x^2 - 5x + 3;$$

and since $x^2 - 5x + 3$ is a factor of both expressions, and is their highest common factor, both expressions will vanish for the values of x given by

$$x^2 - 5x + 3 = 0,$$

and for no other values. Thus the values required are the roots of $x^2 - 5x + 3 = 0$,

and these roots are

$$\frac{5 \pm \sqrt{13}}{2}.$$

EXAMPLES XXXVIII.

Find the roots of the following equations:

1. $x^4-5x^2+4=0.$

2. $\dfrac{x^2-1}{9}+\dfrac{1}{x^2}-1=0.$

3. $x^2+\dfrac{1}{x^2}=a^2+\dfrac{1}{a^2}.$

4. $x^4+\dfrac{1}{x^4}=a^4+\dfrac{1}{a^4}.$

5. $(x^2-x)^2-8(x^2-x)+12=0.$

6. $(x^2+x)^2-22(x^2+x)+40=0.$

7. $(x^2+x)(x^2+x+1)=42.$

8. $\dfrac{x^2}{x+1}+\dfrac{x+1}{x^2}=2.$

9. $\dfrac{x}{x^2+1}+\dfrac{x^2+1}{x}=\dfrac{5}{2}.$

10. $(x^2+x+1)\left(x^2+x-\dfrac{3}{2}\right)+1=0.$

11. $\left(x+\dfrac{1}{x}\right)^2+4\left(x+\dfrac{1}{x}\right)=12.$

12. $2x^2-4x-\sqrt{x^2-2x-3}=9.$

13. $x^2+\sqrt{x^2+3x+7}=5-3x.$

14. $x^2+(x-2)(x-3)+\sqrt{2x^2-5x+6}=6.$

15. $(x+5)(x-2)-36=\sqrt{(x+4)(x-1)}.$

Solve the following cubic equations having given one of the roots of each.

16. $x^3-2x+1=0,\ [x=1].$

17. $x^3-5x+4=0,\ [x=1].$

18. $x^3-49x+120=0,\ [x=3].$

19. $x^3-3x^2-7x+21=0,\ [x=3].$

20. $x^3-2x^2-7x-4=0,\ [x=-1].$

21. $5x^3-15x^2+3x+14=0,\ [x=2].$

22. For what values of x will x^3+2x^2+9 and $x^3-4x+15$ both vanish?

CHAPTER XVI.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE.

150. IN Chapter VIII. we shewed how to solve simultaneous equations of the first degree; we now proceed to consider simultaneous equations of the second degree.

We first take the case of two equations which contain two unknown quantities, one equation being of the first degree and the other of the second degree.

For example, to solve the equations

$$x + 2y = 5,$$

$$x^2 + 2y^2 = 9.$$

From the first equation we have

$$x = 5 - 2y.$$

Substituting this value of x in the second equation, we have

$$(5 - 2y)^2 + 2y^2 = 9,$$

or $$6y^2 - 20y + 16 = 0,$$

or $$3y^2 - 10y + 8 = 0,$$

that is $$(3y - 4)(y - 2) = 0.$$

Hence $$y = \tfrac{4}{3}, \text{ or } y = 2.$$

If $y = \frac{4}{3}$, $x = 5 - 2y = 5 - \frac{8}{3} = \frac{7}{3}$.

And if $y = 2$, $x = 5 - 2y = 5 - 4 = 1$.

Thus the required roots are $x = \frac{7}{3}$, $y = \frac{4}{3}$; or $x = 1$, $y = 2$.

This result should not be written in the form $x = \frac{7}{3}$ or 1, $y = \frac{4}{3}$ or 2.

From the above example it will be seen that to solve two equations, of which one is of the first degree, and the other of the second degree, we proceed as follows:—From the equation of the first degree, find the value of one of the unknown quantities in terms of the other unknown quantity and the known quantities, and substitute this value in the equation of the second degree: one of the unknown quantities is thus eliminated, and a quadratic equation is obtained, the roots of which are the values of the unknown quantity which is retained.

EXAMPLES. XXXIX.

Solve the following equations:

1. $x + y = 6,\quad x^2 - y^2 = 24.$

2. $x - y = 10,\quad x^2 + y^2 = 58.$

3. $3x + 3y = 10,\quad xy = 1.$

4. $2x + 3y = 3,\quad 4x^2 + 9xy + 9y^2 = 44.$

5. $x + y = \frac{15}{4},\quad x - y = xy.$

6. $2x - y = 5,\quad x + 3y = 2xy.$

7. $x + y = 2,\quad \frac{2}{x} + \frac{3}{y} = 6.$

8. $x + 2y = 7,\quad \frac{3}{x} + \frac{6}{y} = 5.$

9. $x + y = 5,\quad \frac{1}{x} + \frac{1}{y} = \frac{5}{6}.$

10. $x - y = 1,\quad \frac{x}{y} - \frac{y}{x} = \frac{5}{6}.$

11. $xy + x = 15,\quad xy - y = 8.$

12. $xy + 2x = 5,\quad 2xy - y = 3.$

151. It should be remarked that we cannot solve *any* two equations which are both of the second degree; for the elimination of one of the unknown quantities will frequently lead to an equation of higher degree than the second, from which the remaining unknown quantity would have to be found; and we cannot solve an equation of higher degree than the second, except in very special cases.

For example, if we have the equations $x^2+x+y=3$ and $x^2+y^2=5$. We have from the first equation $y=3-x-x^2$; and, by substituting this value of y in the second equation,

we get $$x^2+(3-x-x^2)^2=5,$$

that is $$x^4+2x^3-4x^2-6x+4=0;$$

and this equation of the fourth degree cannot be solved by any methods which are within the range of this book.

152. We can always solve two equations of the second degree when all the terms which contain the unknown quantities are of the second degree.

For example, to solve the equations

$$x^2+3xy=28,$$
$$xy+4y^2=\ \ 8.$$

Divide the members of the first equation by the corresponding members of the second equation; we then have

$$\frac{x^2+3xy}{xy+4y^2}=\frac{28}{8}=\frac{7}{2}.$$

Hence $$2x^2+6xy=7xy+28y^2,$$

or $$2x^2-xy-28y^2=0,$$

that is, $$(2x+7y)(x-4y)=0.$$

Hence $$x=4y,\text{ or }x=-\tfrac{7}{2}y.$$

I. If $x = 4y$, we have from the second equation,

$$4y^2 + 4y^2 = 8;$$

$$\therefore\ y = \pm 1.$$

And therefore $$x = 4y = \pm 4.$$

II. If $x = -\frac{7}{2}y$, we have from the second equation,

$$-\tfrac{7}{2}y^2 + 4y^2 = 8;$$

$$y^2 = 16;$$

$$\therefore\ y = \pm 4.$$

And therefore $$x = -\tfrac{7}{2}y = \mp 14.$$

Thus there are four sets of values, namely $x = 4$, $y = 1$; $x = -4$, $y = -1$; $x = 14$, $y = -4$; and $x = -14$, $y = 4$.

We have in the above written down the factors of

$$2x^2 - xy - 28y^2$$

by inspection: when this cannot be done, the factors can be found by the method of Art. 138. Thus in the present example, we should proceed as follows:

$$\begin{aligned} 2x^2 - xy - 28y^2 &= 2\left\{x^2 - \tfrac{1}{2}xy - 14y^2\right\} \\ &= 2\left\{(x - \tfrac{1}{4}y)^2 - (14y^2 + \tfrac{1}{16}y^2)\right\} = 2\left\{(x - \tfrac{1}{4}y)^2 - \tfrac{225}{16}y^2\right\} \\ &= 2\left\{x - \tfrac{1}{4}y + \tfrac{15}{4}y\right\}\left\{x - \tfrac{1}{4}y - \tfrac{15}{4}y\right\} = (2x + 7y)(x - 4y). \end{aligned}$$

153. Any pair of equations which are *homogeneous*, so far as the terms which contain the unknown quantities are concerned, can be solved by the method adopted in the last Article. Sometimes however the equations can be solved more easily by special methods.

For example, to solve

$$x^2 + y^2 = 74,$$

$$2xy = 70.$$

By addition, we have

$$x^2 + 2xy + y^2 = 144,$$

that is $$(x+y)^2-12^2=0;$$

$$\therefore\ (x+y-12)(x+y+12)=0.$$

Hence $$x+y=12 \ldots\ldots\ldots\ldots\text{(i)},$$

or $$x+y=-12 \ldots\ldots\ldots\ldots\text{(ii)}.$$

Again, from the given equations, we have by subtraction,

$$x^2-2xy+y^2=4,$$

that is $$(x-y)^2-2^2=0;$$

$$\therefore\ (x-y-2)(x-y+2)=0.$$

Hence $$x-y=2 \ldots\ldots\ldots\ldots\text{(iii)},$$

or $$x-y=-2 \ldots\ldots\ldots\ldots\text{(iv)}.$$

From (i) and (iii), we have

$$x=7,\ y=5.$$

From (i) and (iv), we have

$$x=5,\ y=7.$$

From (ii) and (iii), we have

$$x=-5,\ y=-7.$$

And, from (ii) and (iv), we have

$$x=-7,\ y=-5.$$

Thus there are four pairs of values, two of which are given by $x=\pm 7$, $y=\pm 5$, and the other two by $x=\pm 5$, $y=\pm 7$, it being understood that in both cases the upper signs are to be taken together, and the lower signs are to be taken together.

Again, taking the equations considered in Art. 152, namely

$$x^2+3xy=28,$$

$$xy+4y^2=\ \ 8.$$

We have by addition

$$x^2+4xy+4y^2=36,$$

that is $(x+2y)^2-6^2=0,$

$$\therefore\ x+2y=6 \ldots\ldots\ldots\ldots\ldots\text{(i)},$$

or $x+2y=-6 \ldots\ldots\ldots\ldots\ldots\text{(ii)}.$

We can now complete the solution as in Art. 150, by taking (i) and (ii) with either of the given equations.

154. The following examples will shew how to deal with some other cases of simultaneous equations.

Ex. 1. Solve $x-y=2 \ldots\ldots\ldots\ldots\ldots\text{(i)},$

$$x^3-y^3=386 \ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) we have $x=y+2.$

Substitute this value in (ii), and we have

$$(y+2)^3-y^3=386,$$

that is $6y^2+12y+8-386=0,$

or $y^2+2y-63=0.$

Hence $y=7,$ or $y=-9.$

If $y=7,$ $x=y+2=9.$

If $y=-9,$ $x=y+2=-7.$

Thus $x=9,\ y=7,$ or $x=-7,\ y=-9.$

Ex. 2. Solve $x-3y=2,$

$$x^2-9y^2=8.$$

Divide the members of the second equation by the corresponding members of the first; then

$$x+3y=4;$$

and from $x+3y=4$ and $x-3y=2,$

we have $x=3,\ y=\frac{1}{3}.$

Ex. 3. Solve $x^2-xy+y^2=61 \ldots\ldots\ldots\ldots\text{(i)},$

$$x^4+x^2y^2+y^4=1281\ldots\ldots\ \ldots\ldots\text{(ii)}.$$

Divide the members of (ii) by the corresponding members of (i); we then have

$$x^2+xy+y^2=21\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

From (i) and (iii) we have, by subtraction,

$$2xy = -40,$$

or $$xy = -20 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

From (i) and (iv), we have

$$x^2 - 2xy + y^2 = 81;$$

$$\therefore\ x - y = +9 \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

or $$x - y = -9 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

From (i) and (iv), we have

$$x^2 + 2xy + y^2 = 1;$$

$$\therefore\ x + y = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)},$$

or $$x + y = -1 \ \ldots\ldots\ldots\ldots\ldots\text{(viii)}.$$

Then combining either of the equations (v) and (vi) with either of the equations (vii) and (viii) we get the four pairs of values $x = \pm 5,\ y = \mp 4$; $x = \pm 4,\ y = \mp 5$.

Ex. 4. Solve $$x^2 - 2xy = 3y \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$2x^2 - 9y^2 = 9y \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Multiply both sides of (i) by 3; then

$$3x^2 - 6xy = 9y \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Hence $$2x^2 - 9y^2 = 3x^2 - 6xy;$$

$$\therefore\ x^2 - 6xy + 9y^2 = 0,$$

that is $$(x - 3y)^2 = 0;$$

$$\therefore\ x = 3y.$$

Substitute in (ii); then

$$2(3y)^2 - 9y^2 = 9y,$$

or $$9y^2 - 9y = 0;$$

$$\therefore\ y(y - 1) = 0.$$

Hence $$y = 0, \text{ or } y = 1,$$

If $y = 0$, $\quad x = 3y = 0.$

If $y = 1$, $\quad x = 3y = 3.$

Thus $\quad x = 0,\ y = 0;$

or else $\quad x = 3,\ y = 1.$

Ex. 5. Solve

$$xy + xz = 27 \quad \ldots\ldots\ldots\ldots\text{(i)},$$
$$yz + yx = 32 \quad \ldots\ldots\ldots\ldots\text{(ii)},$$
$$zx + zy = 35 \quad \ldots\ldots\ldots\ldots\text{(iii)}.$$

From (i) and (ii), we have by addition

$$2xy + xz + yz = 59 \quad \ldots\ldots\ldots\ldots\text{(iv)}.$$

From (iv) and (iii), we have by subtraction

$$2xy = 24;$$
$$\therefore\ xy = 12 \quad \ldots\ldots\ldots\ldots\text{(v)}.$$

Hence, from (i), $xz = 15 \quad \ldots\ldots\ldots\ldots\text{(vi)}.$

And, from (ii), $yz = 32 - 12 = 20 \quad \ldots\ldots\ldots\ldots\text{(vii)}.$

From (v) and (vi), we have by multiplication

$$x^2yz = 180 \quad \ldots\ldots\ldots\ldots\text{(viii)}.$$

From (vii) and (viii) we have by division

$$x^2 = 180 \div 20 = 9;$$
$$\therefore\ x = \pm 3.$$

Hence, from (v), $y = 12 \div (\pm 3) = \pm 4.$

And, from (vi), $z = 15 \div (\pm 3) = \pm 5.$

Thus $x = \pm 3,\ y = \pm 4,\ z = \pm 5.$

All the upper signs being taken together, and all the lower signs being taken together.

EXAMPLES. XL.

Solve the following equations:

1. $x^2 - 2xy = 0,$
$4x^2 + 9y^2 = 225.$

2. $2x^2 = 3xy,$
$y^2 + 5xy = 34.$

3. $x^2 + xy = 21,$
$y^2 + xy = 28.$

4. $x^2 + xy = (a - b)^2,$
$xy + y^2 = 4ab.$

5. $x^2 + xy = 24,$
$2y^2 + 3xy = 32.$

6. $x^2 - 3xy = 10,$
$4y^2 - xy = -1.$

7. $x^2 + xy - 2y^2 = -44.$
$xy + 3y^2 = 80.$

8. $x^2 + 3xy = 7,$
$y^2 + xy = 6.$

9. $x^2 - 3y^2 = 13,$
$3x^2 - y^2 = 71.$

10. $3xy + x^2 = 10,$
$5xy - 2x^2 = 2.$

11. $$\begin{aligned} x^2+3xy&=54,\\ xy+4y^2&=115. \end{aligned}$$

12. $$\begin{aligned} x(x+y)&=40,\\ y(x-y)&=6. \end{aligned}$$

13. $$\begin{aligned} x^2-3xy+2y^2&=3,\\ 2x^2+y^2&=6. \end{aligned}$$

14. $$\begin{aligned} x^2-xy&=10,\\ (x-y)^2&=4. \end{aligned}$$

15. $$\begin{aligned} x+y&=-1,\\ \frac{1}{x}+\frac{1}{y}&=\frac{1}{12}. \end{aligned}$$

16. $$\begin{aligned} x+y&=-3,\\ \frac{1}{x}+\frac{1}{y}&=\frac{1}{6}. \end{aligned}$$

17. $$\begin{aligned} \frac{x}{3}+\frac{y}{2}&=\frac{5}{3},\\ \frac{2}{x}+\frac{3}{y}&=0. \end{aligned}$$

18. $$\begin{aligned} \frac{x}{2}+\frac{y}{5}&=5,\\ \frac{2}{x}+\frac{5}{y}&=\frac{5}{6}. \end{aligned}$$

19. $$\begin{aligned} (x+1)(y+1)&=10,\\ xy&=3. \end{aligned}$$

20. $$\begin{aligned} (x+3)(y+1)&=4,\\ xy&=-1. \end{aligned}$$

21. $$\begin{aligned} x-y&=3,\\ x^3-y^3&=279. \end{aligned}$$

22. $$\begin{aligned} x-y&=2,\\ x^3-y^3&=98. \end{aligned}$$

23. $$\begin{aligned} x+y&=7,\\ x^3+y^3&=91. \end{aligned}$$

24. $$\begin{aligned} x+y&=1,\\ x^3+y^3&=61. \end{aligned}$$

25. $$\begin{aligned} x^2+xy+y^2&=13,\\ x^4+x^2y^2+y^4&=91. \end{aligned}$$

26. $$\begin{aligned} x^2-xy+y^2&=9,\\ x^4+x^2y^2+y^4&=243. \end{aligned}$$

27. $$\begin{aligned} x+y&=1,\\ x^2y^2+13xy+12&=0. \end{aligned}$$

28. $$\begin{aligned} x+y&=5,\\ 4xy&=12-x^2y^2. \end{aligned}$$

29. $$\begin{aligned} x+\frac{1}{y}&=3,\\ y+\frac{1}{x}&=\frac{4}{3}. \end{aligned}$$

30. $$\begin{aligned} x+\frac{1}{y}&=\frac{18}{7},\\ y+\frac{1}{x}&=\frac{7}{4}. \end{aligned}$$

31. $$\begin{aligned} 2x^2-xy+y^2&=2y,\\ 2x^2+4xy&=5y. \end{aligned}$$

32. $$\begin{aligned} x^3+1&=9y,\\ x^2+x&=6y. \end{aligned}$$

33. $$\begin{aligned} 2x^3+5y^3&=115,\\ 3x^3+7y^3&=186. \end{aligned}$$

34. $$\begin{aligned} x^2y+xy^2&=180,\\ x^3+y^3&=189. \end{aligned}$$

35. $$\begin{aligned} xy+\frac{x}{y}&=\frac{5}{3},\\ xy+\frac{y}{x}&=\frac{5}{6}. \end{aligned}$$

36. $$\begin{aligned} a(x-a)&=b(y-b),\\ xy&=ax+by. \end{aligned}$$

37. $$x-\frac{b^2}{y}=\frac{a^2}{x}-y=a-b.$$

CHAPTER XVII.

PROBLEMS.

155. WE now give examples of problems in which the relations between the known and unknown quantities are expressed algebraically by means of quadratic equations.

In the solution of problems it often happens that the roots of the equation, which is the algebraical statement of the relation between the magnitudes of the known and unknown quantities, do not all satisfy the conditions of the problem.

The reason of this is that the roots of the equation are the numbers, whether *positive or negative*, *integral or fractional*, which satisfy it; but in the problem itself there may be restrictions, expressed or implied, on the numbers, and these restrictions cannot be retained in the equation. For example, if one of the roots of an equation is fractional, it cannot be a solution of a problem which refers to a number of men, for such a number must be *integral.*

The following are examples of problems which lead to quadratic equations.

Ex. 1. How many children are there in a family, when eleven times the number is greater by five than twice the square of the number?

Let x be the number of children; then we have

$$11x = 2x^2 + 5,$$

or $$2x^2 - 11x + 5 = 0,$$

that is $$(2x - 1)(x - 5) = 0.$$

Hence $$x = 5, \text{ or } x = \tfrac{1}{2}.$$

Thus there are 5 children, the value $\frac{1}{2}$ being inadmissible.

Ex. 2. Eleven times the number of yards in the length of a rod is greater by five than twice the square of the number of yards. How long is the rod?

This leads to the same equation as before, only in this case we cannot reject the fractional result. Thus the rod may be five yards long, or it may be half a yard long.

Ex. 3. A number of two digits is equal to twice the product of the digits, and the digit in the ten's place is less by 3 than the digit in the unit's place. What is the number?

Let x be the digit in the ten's place; then $x + 3$ will be the digit in the unit's place. The number is therefore equal to $$10x + (x + 3).$$

Hence, by the question,

$$10x + x + 3 = 2x(x + 3);$$

$$\therefore\ 2x^2 - 5x - 3 = 0,$$

that is $$(x - 3)(2x + 1) = 0.$$

Hence $$x = 3, \text{ or } x = -\tfrac{1}{2}.$$

Now the digits of a number must be *positive integers;* hence the second value is inadmissible.

Therefore the digits are 3 and 6, and the required number is 36.

Ex. 4. The square of the number of pounds a man possesses is greater by 1000 than thirty times the number. How much is the man worth?

Let x be the number of pounds the man is worth; then, by the question,

$$x^2 = 30x + 1000;$$

$$\therefore\ x^2 - 30x - 1000 = 0,$$

that is

$$(x - 50)(x + 20) = 0.$$

Hence

$$x = 50, \text{ or } x = -20.$$

Both these values are admissible, since a negative possession is a debt.

Hence the man may have £50, or he may owe £20.

Ex. 5. The sum of a certain number and its square root is 42: what is the number?

Let x be the number; then

$$x + \sqrt{x} = 42,$$

or

$$\sqrt{x} = 42 - x.$$

Square both sides; then, after transposition, we have

$$x^2 - 85x + 1764 = 0,$$

the roots of which are 36 and 49.

The value 49 will not however satisfy the condition of the question, if by the square root of the number is meant only the *arithmetical* square root.

Ex. 6. The sum of the ages of a father and his son is 100 years; also one-tenth of the product of their ages, in years, exceeds the father's age by 180. How old are they?

Let the father be x years old; then the son will be $100 - x$ years old. Hence, by the question,

$$\tfrac{1}{10}x(100 - x) = x + 180;$$

$$\therefore\ x^2 - 90x + 1800 = 0,$$

that is $$(x-60)(x-30)=0.$$

Hence $$x=60, \text{ or } x=30.$$

The second value is inadmissible, although it is a positive integer, for it would make the son older than his father.

Hence the father must be 60, and the son 40 years old.

EXAMPLES. XLI.

1. Find two numbers one of which is three times the other and whose product is 243.

2. Find two numbers whose sum is 18 and whose product is 77.

3. Find two numbers whose difference is 20, and the sum of whose squares is 650.

4. Divide 25 into two parts whose product is 156.

5. Divide 80 into two parts the sum of the squares of which is 3208.

6. A certain number is subtracted from 36, and the same number is also subtracted from 30; and the product of the remainders is 891. What is the number?

7. A rectangular court is ten yards longer than it is broad; its area is 1131 square yards. What is its length and breadth?

8. The product of the sum and difference of a number and its reciprocal number is $3\frac{3}{4}$: find the number.

9. The number of fives' balls which can be bought for a pound is equal to the number of shillings in the cost of 125 of them. How many can be bought for a pound?

10. The number of eggs which can be bought for one shilling is equal to the number of pence which 27 eggs cost. How many eggs can be bought for one shilling?

11. A cask contains a certain number of gallons of water, and another cask contains half as many gallons of wine; six gallons are drawn from each, and what is drawn from the one cask is put into the other. If the mixture in each cask be now of the same strength, find the amount of water and wine.

12. The cost of an entertainment was £4, which was to have been divided equally among the party, but four of them leave without paying, and the rest have each to pay one shilling extra in consequence. Of how many persons did the party consist?

13. A man buys a certain number of articles for £1, and makes £1. 1*s*. 0*d*. by selling all but two at 2*d*. a piece more than they cost. How many did he buy?

14. A man bought a certain number of railway-shares for £1875; he sold all but 15 of them for £1740, gaining £4 per share on their cost price; how many shares did he buy?

15. A crew can row a certain course up stream in $8\frac{4}{7}$ minutes, and if there were no stream they could row it in 7 minutes less than it takes them to drift down the stream; how long would they take to row down with the stream?

16. A boat's crew can row 8 miles an hour in still water. What is the speed of a river's current if it take them 2 hours and 40 minutes to row 8 miles up and 8 miles down?

17. Two trains run without stopping over the same 36 miles of rail. One of them travels 15 miles an hour faster than the other, and accomplishes the distance in 12 minutes less. Find the speed of the two trains.

18. A person having 7 miles to walk increases his speed one mile an hour after the first mile, and is half an hour less on the road than he would have been had he not altered his rate. How long did he take?

19. *A* and *B* together can do a piece of work in a certain time. If they each did one-half of the work separately, *A* would have to work one day less, and *B* two days more than before. Find the time in which *A* and *B* together do the work.

20. The price of photographs is raised 3*s*. per dozen; and, in consequence, seven less than before are sold for a guinea. What was the original price?

21. What are eggs a dozen when two more for a shilling would lower the price a penny a dozen?

22. A woman spends three shillings in eggs; if she had bought a dozen less for the same money, they would have cost her threepence a dozen more. How many did she buy?

23. The price of one kind of sugar per stone is 1*s.* 9*d.* more than that of another kind, and 8 lbs. less of the first kind can be got for £1 than of the second. Find the price of each per stone.

24. The number of pence which a dozen apples cost is greater by 2 than twice the number of apples which can be bought for 1*s.* How many can be bought for 9*s.*?

25. Divide £1015 among A, B and C so that B shall receive £5 less than A, and C as many times B's share as there are shillings in A's share.

26. Find two fractions whose sum is $\frac{5}{6}$, and whose difference is equal to their product.

27. Two men start at the same time to meet each other from towns which are 25 miles apart. One takes 18 minutes longer than the other to walk a mile, and they meet in 5 hours. How fast does each walk?

28. The men in a regiment can be arranged in a column twice as deep as it is broad. If the number be diminished by 206, the men can be arranged in a hollow square three deep, having the same number of men in each outer side of the square as there were in the depth of the column. How many men were there at first in the regiment?

29. The area of a certain rectangle is equal to the area of a square whose side is six inches shorter than one of the sides of the rectangle. If the breadth of the rectangle be increased by one inch and its length diminished by two inches, its area is unaltered. Find the lengths of its sides.

30. The diagonal and the longer side of a rectangle are together five times the shorter side, and the longer side exceeds the shorter by 35 yards. What is the area of the rectangle?

31. If the greater side of a rectangle be diminished by 3 yards and the less by 1 yard its area would be halved; and if the greater be increased by 9 and the less diminished by 2 its area would be unaltered. Find the sides.

32. Two trains A and B leave P for Q at the same time as two trains C and D leave Q for P. A passes C 120 miles from P, and D 140 miles from P. B passes C 126 miles from Q, and D half way between P and Q: find the distance from P to Q.

MISCELLANEOUS EXAMPLES. IV.

A.

1. Simplify $2x-[3x-9y-\{2x-3y-(x+5y)\}]$.

2. Multiply $a^2+25b^2+4c^2+5ab-2ac+10bc$ by $a-5b+2c$.

3. Divide $x^3+(4ab-b^2)x-(a-2b)(a^2+3b^2)$ by $x-a+2b$.

4. Find the factors of (i) $(2x+y-z)^2-(x+2y+4z)^2$.
   (ii) $x^2y^2-x^2-y^2+1$, and (iii) $x^2y^2z^2-x^2z-y^2z+1$.

5. Simplify (i) $\dfrac{x^3+4x^2-8x+24}{8-8x+x^3-x^4}$.

   (ii) $\dfrac{x-1}{(x-2)(x-3)}-2\dfrac{x-2}{(x-3)(x-1)}+\dfrac{x-3}{(x-1)(x-2)}$.

6. Solve the equations:

   (i) $\dfrac{1}{x-a}+\dfrac{2}{x-b}=\dfrac{3}{x-c}$.

   (ii) $4x^2-25x-21=0$.

   (iii) $x+y=\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{2}$.

7. Shew that x^2-5x+7 can never be less than $\frac{3}{4}$.

8. The difference of the cubes of two consecutive numbers is 919: find the numbers.

B.

1. Simplify $(x+3)^3-3(x+2)^3+3(x+1)^3-x^3$.

2. Shew that

$$(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=(xa+by-1)^2+(bx-ay)^2.$$

3. Divide $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.

4. Find the L.C.M. of $8x^3+27$, $16x^4+36x^2+81$ and $6x^2-5x-6$.

5. Simplify

   (i) $\dfrac{2}{x-1}-\dfrac{1}{x+1}-\dfrac{x+3}{x^2+1}$.

   (ii) $\dfrac{\{(a+b)(a+b+c)+c^2\}\{(a+b)^2-c^2\}}{\{(a+b)^3-c^3\}(a+b+c)}$.

6. Solve the equations:

(i) $(6-x)(1+2x)+3x(x+5)=(x+1)^2-x.$

(ii) $5x^2+7x=160.$

(iii) $x^2+xy=10,\ y^2-xy=3.$

7. Shew that the sum of the squares of the roots of the equation $x^2-5x+2=0$ is 21.

8. At a concert £60 was received for reserved seats, and the same sum for unreserved seats. A reserved seat cost 3 shillings more than an unreserved seat, but 360 more tickets for unreserved than for reserved seats were sold. How many tickets were sold altogether?

C.

1. Simplify $12a-3\{b-2(a-3b)-2a\}.$

2. Multiply $a^3+2a^2b-ab^2+2b^3$ by $a^3-2a^2b-ab^2-2b^3.$

3. Divide $24x^4-10x^3y-8x^2y^2+10xy^3-4y^4$ by $2y^2-xy-4x^2.$

4. Find the factors of $9x^2+9x+2,$

and of $4(ab-cd)^2-(a^2+b^2-c^2-d^2)^2.$

5. Simplify

$$\frac{x+\dfrac{y-x}{1+xy}}{1-\dfrac{x(y-x)}{1+xy}},$$

and shew that

$$\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}+\frac{(b-c)(c-a)(a-b)}{abc}=0.$$

6. Solve the equations:

(i) $\dfrac{x-1}{3}-\dfrac{x-3}{5}=7+\dfrac{x+2}{6}.$

(ii) $x+y=2a,\ x^2+y^2=2a^2.$

7. Find the least value of $x^2+6x+12$, and the greatest value of $6x-x^2-4$.

8. *A* and *B* have eighteen coins between them, which are all shillings or pennies. *A* has three times as many shillings as pennies, and *B* has just as many shillings as pennies; also *A* has 9*d.* more than *B*. How much has each?

D.

1. Shew that $(b+c)^2-a^2+(c+a)^2-b^2+(a+b)^2-c^2=(a+b+c)^2$.

2. Arrange $(1+x)^4+2(1-x+x^2)$ according to ascending powers of x.

3. Shew that the difference between the squares of any two consecutive numbers is one more than double the smaller number.

4. Shew that if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, $a^2+b^2+c^2=(a+b+c)^2$.

5. Find the L. C. M. of

$$x^2-5x-14,\ x^2-4x-21 \text{ and } x^3-3x^2-25x-21.$$

For what value of x will all three expressions vanish?

6. Solve the equations:

(i) $\frac{4}{x}-\frac{3}{y}+5=\frac{6}{x}+\frac{3}{y}=10$.

(ii) $\frac{1}{x}-\frac{1}{a}-\frac{1}{b}=\frac{1}{x-a-b}$.

(iii) $x+y=3$, $3x^2-11y^2=1$.

7. If x_1, x_2 are the roots of $ax^2+bx+c=0$, prove that

$$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{b^2-2ac}{ac}.$$

8. Out of a cask containing 60 gallons of alcohol a certain quantity is drawn off and replaced by water. Of the mixture a second quantity, 14 gallons more than the first, is drawn off and replaced by water. The cask then contains as much water as alcohol. How much was drawn off the first time?

E.

1. Find the numerical value of

$$\frac{x}{2}-\left\{\frac{2x-3}{3}-\frac{3x-1}{4}\right\}\div\frac{x-1}{2}\text{, when } x=4\tfrac{1}{3}.$$

2. Shew that $(a+b)^4-(a^2-b^2)^2=4ab(a+b)^2$, and that

$$2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b)$$
$$=(b-c)^2+(c-a)^2+(a-b)^2.$$

3. Divide $(a+2b-3c+d)^2-(2a+b+3c-d)^2$ by $a+b$.

4. Find the L. C. M. of $6x^2-5ax-6a^2$ and $4x^3-2ax^2-9a^3$.

5. Simplify $\left(\frac{y^2}{x}+\frac{x^2}{y}\right)\frac{1}{y^2-x^2}+\frac{x}{xy-y^2}-\frac{y}{x^2+xy}$.

6. Solve the equations:

(i) $\frac{x+a}{x-b}+\frac{x+b}{x-a}=2.$

(ii) $(x-1)(x-2)+(x-2)(x-3)+(x-3)(x-1)=11.$

(iii) $x+\frac{1}{y}=\frac{21}{10}, \qquad y+\frac{1}{x}=\frac{7}{3}.$

7. For what values of x are $\frac{x}{a}+\frac{a}{x}$ and $\frac{c}{a}+\frac{a}{c}$ equal to one another?

8. A tricyclist rode 180 miles at a uniform rate. If he had ridden 3 miles an hour slower than he did, it would have taken him 3 hours longer. How many miles an hour did he ride?

F.

1. Add together $3a^3-4a^2b+2ab^2$, $3a^2b-4ab^2+2b^3$, $3ab^2-4b^3+2a^3$ and $3b^3-4a^3+2a^2b$.

2. Shew that
$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$$
$$=(ay-bx)^2+(bz-cy)^2+(cx-az)^2.$$

3. Divide $2+11x+11x^2+x^3-x^4$ by x^2+3x+2.

4. Find the H.C.F. of x^3-x^2-2x+2 and $x^4-3x^3+2x^2+x-1$. What value of x will make both expressions vanish?

5. Simplify (i) $\frac{2}{x(x-2)}+\frac{1}{x^2-5x+6}+\frac{3}{x(3-x)}.$

(ii) $$\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-x}}}.$$

6. Solve the equations:

(i) $\frac{x+3}{5}-\frac{6-x}{10}=x-\frac{7}{10}.$

(ii) $\frac{x+y}{a+b}+\frac{x-y}{a-b}=2,\ ax+by=a^2+b^2.$

(iii) $\frac{x+1}{x-1}+\frac{x+2}{x-2}=2\frac{x+3}{x-3}.$

7. Find the difference of the squares of the roots of the equation
$$x^2-7x+9=0.$$

8. What number exceeds its square root by 156?

CHAPTER XVIII.

POWERS AND ROOTS.

156. THE process by which the powers of quantities are obtained is often called *involution;* and the reverse process, namely that by which the roots of quantities are obtained, is called *evolution.*

We shall in the present chapter consider some cases of involution, and of evolution.

157. We have proved in Art. 32 that when m and n are any positive integers,

$$a^m \times a^n = a^{m+n}.$$

Thus the index of the product of any two powers of the same quantity is the *sum* of the indices of the factors.

The same law holds good for the product of any number of powers of the same quantity. For we have

$$a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p},$$

and so on, however many factors there may be.

Hence $$a^m \times a^n \times a^p \ldots\ldots = a^{m+n+p+\cdots}.$$

Thus, *the index of the product of any number of powers of the same quantity is the sum of the indices of the factors.*

158. By the Index Law

$$a^{m-n} \times a^n = a^{m-n+n}$$
$$= a^m.$$

Hence $$a^m \div a^n = a^{m-n}.$$

The above result can also be obtained as follows:

$$a^m \div a^n = \frac{a \times a \times a \times a \ldots \text{ to } m \text{ factors}}{a \times a \times a \times a \ldots \text{ to } n \text{ factors}}.$$

Now, if m is greater than n, the n factors of the denominator can be cancelled with n of the factors of the numerator: we then have $m - n$ factors left in the numerator.

Thus, when m is greater than n,

$$a^m \div a^n = a^{m-n}.$$

If however n is greater than m, the m factors of the numerator can be cancelled with m of the factors of the denominator: we then have $n - m$ factors left in the denominator.

Thus, when m is less than n,

$$a^m \div a^n = \frac{1}{a^{n-m}}.$$

159. To find $(a^m)^n$ when m and n are positive integers.

By definition

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots \text{ to } n \text{ factors}\\ &= a^{m+m+m+\ldots \text{ to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

Hence $$(a^m)^n = a^{mn}.$$

Thus, *to raise any power of a quantity to any other power, its original index must be multiplied by the index of the power to which it is to be raised.*

160. To find $(ab)^m$.

$$\begin{aligned}(ab)^m &= ab \times ab \times ab \ldots\ldots \text{ to } m \text{ factors}, && \text{by definition}\\ &= (aaa \ldots \text{ to } m \text{ factors}) \times (bbb \ldots \text{ to } m \text{ factors}) && \text{[Art. 31.]}\\ &= a^m \times b^m, && \text{by definition.}\end{aligned}$$

Hence $$(ab)^m = a^m b^m.$$

Similarly

$$\begin{aligned}(abc)^m &= abc \times abc \times abc \ldots \text{ to } m \text{ factors,}\\ &= (aaa \ldots \text{ to } m \text{ factors}) \times (bbb \ldots \text{ to } m \text{ factors})\\ &\qquad\qquad \times (ccc \ldots \text{ to } m \text{ factors})\\ &= a^m \times b^m \times c^m.\end{aligned}$$

Hence $(abc)^m = a^m b^m c^m$,

and so on, however many factors there may be in the expression whose mth power is required.

Thus, *the* m*th power of* a *product is the product of the* m*th powers of its factors.*

161. We can now consider the most general case of the involution of a *simple* expression.

To find $(a^x b^y c^z \ldots)^m$.

By Art. 160, $(a^x b^y c^z \ldots)^m$

$$\begin{aligned}&= (a^x)^m (b^y)^m (c^z)^m \ldots\\ &= a^{xm} b^{ym} c^{zm} \ldots \qquad\qquad [\text{Art. } 159.]\end{aligned}$$

Thus any power of a simple expression is obtained by taking each of its factors to a power whose index is the product of its original index and the index of the power to which the whole expression is to be raised.

162. The following is an important case.

To find $\left(\frac{a}{b}\right)^m$.

$$\begin{aligned}\left(\frac{a}{b}\right)^m &= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \ldots\ldots \text{ to } m \text{ factors, by definition,}\\ &= \frac{a \times a \times a \ldots \text{ to } m \text{ factors}}{b \times b \times b \ldots \text{ to } m \text{ factors}} \ldots\ldots\ldots\ldots [\text{Art. } 130.]\\ &= \frac{a^m}{b^m}.\end{aligned}$$

$$\therefore \quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

163. It should be noticed that all powers of a positive quantity are positive, and that successive powers of a negative quantity are alternately positive and negative. This follows at once from the Law of Signs; for we have

$$(-a)^2=(-a)(-a)=+a^2;$$
$$(-a)^3=(-a)^2(-a)=(+a^2)(-a)=-a^3;$$
$$(-a)^4=(-a)^3(-a)=(-a^3)(-a)=+a^4;$$

and so on.

Thus $\quad (-a)^{2n}=+a^{2n}$, and $(-a)^{2n+1}=-a^{2n+1}$.

From the above it is clear that all even powers, whether of positive or of negative quantities, are positive; and that all odd powers of any quantity have the same sign as the original quantity.

164. We have already proved the following cases of the involution of binomial expressions.

$$(a+b)^2=a^2+2ab+b^2,$$

and
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

If we multiply again by $a+b$, we shall have

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

By multiplying the last result by $a+b$ we should obtain $(a+b)^5$, and by continuing the process we could obtain any required power of $a+b$; but to find in this way any high power, for instance to find $(a+b)^{20}$, would clearly be very laborious.

We shall in a subsequent chapter prove a theorem, called the Binomial Theorem, which will enable us to write down at once any power of a binomial expression.

Since the above formulæ are true for *all* values of a and b, we can write down the squares and the cubes of any binomial expressions. Thus

$$\begin{aligned}(a^4-b^4)^2 &= (a^4)^2+2(a^4)(-b^4)+(-b^4)^2\\ &= a^8 \quad -2a^4b^4 \qquad +b^8,\end{aligned}$$

and

$$\begin{aligned}(5a^2-3b^2)^3 &= (5a^2)^3+3(5a^2)^2(-3b^2)\\ &\qquad +3(5a^2)(-3b^2)^2+(-3b^2)^3\\ &= 125a^6-225a^4b^2+135a^2b^4-27b^6.\end{aligned}$$

Also

$$\begin{aligned}(a+b+c)^3 &= \{a+(b+c)\}^3\\ &= a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3.\end{aligned}$$

165. An important case of involution is considered in Art. 67, where the square of a multinomial expression is obtained.

EXAMPLES. XLII.

Write down the value of each of the following:—

1. $(a^3)^5$.
2. $(a^5)^3$.
3. $(-a^2)^3$.
4. $(-a^3)^2$
5. $(-2a^5)^4$.
6. $(-a^4)^5$.
7. $(-ab^2)^4$.
8. $(a^3b^4)^5$.
9. $(-ab^4)^7$.
10. $(-3a^7b^5c)^3$.
11. $(-ab^2c^5)^4$.
12. $(-5a^2b^3c^4)^3$.
13. $\left(\frac{a^2}{b^2}\right)^3$.
14. $\left(-\frac{a^3}{bc^2}\right)^5$.
15. $\left(-\frac{a}{b^2c^3}\right)^6$.
16. $(2a^4+3b^3)^2$.
17. $(a^5-2b^4)^2$.
18. $(-ax^4+by^3)^2$.
19. $(-a^2+2ab)^2$.
20. $(a^2+b^2)^3$.
21. $(a^3+b^3)^3$.
22. $(2a^2-3b^2)^3$.
23. $(3a^2-2b^2)^3$.
24. $(a^2+b^2+c^2)^2$.
25. $(a^3-2b^3+3c^3)^2$.
26. $(a+2b+3c+4d)^2$.
27. $(1+x+x^2+x^3)^2$.

166. We shall now consider some cases of *evolution*.

We know that there are *two* square roots of a^2, namely $\pm a$; we also know [Art. 149] that there are *three* cube roots of a^3, of which a is one, and the other two are imaginary.

There is therefore an important difference between powers and roots; for there is only one nth *power*, but there is more than one nth *root*, of a given expression.

The two square roots of an expression only differ in sign, and either of them is often called *the* square root.

By the cube root of an expression we shall in the present chapter only mean the *real* cube root, and not either of the two imaginary cube roots; and similarly for roots higher than the third.

167. An expression which when raised to the nth power, where n is any positive integer, becomes equal to a given expression, is called *an* nth *root* of the given expression.

We have shewn in Art. 160 that the mth power of a product is the product of the mth powers of its factors; hence, conversely, the mth root of a product is the product of the mth roots of its factors.

Thus $$\sqrt{abc} = \sqrt{a}\,\sqrt{b}\,\sqrt{c},$$

and $$\sqrt[m]{ab} = \sqrt[m]{a}\,\sqrt[m]{b}.$$

Again, we have shewn in Art. 161 that the nth *power* of a simple expression is obtained by *multiplying* the index of each of its factors by n.

It follows conversely that if we *divide* the index of each factor of a given expression by n we shall obtain an nth *root* of the expression. For by raising to the nth power the result obtained by such division of the indices, we must clearly get the original expression.

Thus

$$\sqrt[2]{a^4} = a^2, \qquad \sqrt{\frac{a^2}{b^2}} = \frac{a}{b}, \qquad \sqrt[2]{a^4c^2b^6} = \sqrt[2]{a^4} . \sqrt{c^2} . \sqrt[2]{b^6} = a^2cb^3,$$

and $$\sqrt{a^{np}b^{nq}} = a^pb^q.$$

When the square root of an expression which is not a perfect square, or the cube root of an expression which is not a perfect cube, is required, the operation cannot be performed. We can, for example, only write the square root of a as $\sqrt{a}$, and the cube root of a^2 as $\sqrt[3]{a^2}$, and similarly in other cases.

SQUARE ROOT.

168. We now proceed to consider the square root of compound expressions.

In Art. 95 we have shewn how to write down the square root of any *trinomial* expression which is a complete square.

Having arranged the expression according to ascending or descending powers of some letter, the square root of the whole expression is then found by taking the square roots of the extreme terms with the same or with different signs according as the sign of the middle term is positive or negative.

Thus, to find the square root of

$$4a^8 - 12a^4b^3 + 9b^6.$$

The square roots of the extreme terms are

$$2a^4 \text{ and } 3b^3.$$

Hence, the middle term being negative, the required square root is $$2a^4 - 3b^3.$$

If *both* square roots are required, they are written in the form $$\pm(2a^4 - 3b^3).$$

As other examples

$$\sqrt{(49a^{10} + 28a^5 + 4)} = 7a^5 + 2,$$

$$\sqrt{(1 + 5xy^3 + \tfrac{25}{4}x^2y^6)} = 1 + \tfrac{5}{2}xy^3,$$

and $$\sqrt{\{a^2 + 2a(b + c) + (b + c)^2\}} = a + b + c.$$

169. When an expression which contains only two different powers of a particular letter is arranged according to ascending or descending powers of that letter, it will only contain *three* terms. For example, the expression

$$a^2 + b^2 + c^2 + 2bc + 2ca + 2ab,$$

which contains no other power of a but a^2 and a, when arranged according to powers of a, is

$$a^2 + 2a(b + c) + (b^2 + c^2 + 2bc).$$

Thus any expression which only contains two different powers of a particular letter can be written as a *trinomial* expression; and since we can write down the square root of *any* trinomial expression which is a complete square, it follows that we can write down the square root of any expression which is a complete square, provided that the expression only contains two different powers of some particular letter.

For example, to find the square root of

$$a^2 + b^2 + c^2 + 2bc + 2ac + 2ab.$$

Arranging the expression according to descending powers of a, we have $$a^2 + 2a(b + c) + (b^2 + 2bc + c^2),$$

that is, $$a^2 + 2a(b + c) + (b + c)^2,$$

which is $$\{a + (b + c)\}^2.$$

Thus $$\sqrt{(a^2 + b^2 + c^2 + 2bc + 2ca + 2ab)} = \pm(a + b + c).$$

Also, to find the square root of

$$x^4+4y^4+9z^4+4x^2y^2-6x^2z^2-12y^2z^2.$$

Arrange the expression according to descending powers of x; we then have

$$x^4+2x^2(2y^2-3z^2)+4y^4+9z^4-12y^2z^2,$$

that is, $$x^4+2x^2(2y^2-3z^2)+(2y^2-3z^2)^2,$$

which is $$\{x^2+(2y^2-3z^2)\}^2.$$

Thus $\sqrt{}(x^4+4y^4+9z^4+4x^2y^2-6x^2z^2-12y^2z^2)$
$$=\pm(x^2+2y^2-3z^2).$$

Again, to find the square root of

$$a^2+2abx+(b^2+2ac)x^2+2bcx^3+c^2x^4.$$

Arrange the expression according to powers of a; we then have $$a^2+2a(bx+cx^2)+b^2x^2+2bcx^3+c^2x^4,$$

that is, $$a^2+2a(bx+cx^2)+(bx+cx^2)^2,$$

which is $$\{a+(bx+cx^2)\}^2.$$

Hence the required square root is

$$\pm(a+bx+cx^2).$$

And to find the square root of

$$x^6-2x^5+3x^4+2x^3(y-1)+x^2(1-2y)+2xy+y^2.$$

The expression only contains y^2 and y; we therefore arrange it according to powers of y; we then have

$$y^2+2y(x^3-x^2+x)+x^6-2x^5+3x^4-2x^3+x^2.$$

Now if the expression is a complete square at all, the last of the three terms must be the square of half the coefficient of y; and it is easy to verify that

$$x^6-2x^5+3x^4-2x^3+x^2 \text{ is } (x^3-x^2+x)^2.$$

Thus the given expression is

$$y^2+2y(x^3-x^2+x)+(x^3-x^2+x)^2.$$

The required square root is therefore

$$y+x^3-x^2+x.$$

From the above it will be seen that however many terms there may be in an expression which is a perfect square, the square root can be written down *by inspection*, provided only that the expression contains only *two different powers of some particular letter.*

170. In order to shew how to find the square root of *any* algebraical expression, we will take an expression and form its square, and then shew how to reverse the process.

Consider, for example,

$$x^2 + 2xy + 3y^2 \text{.......................(i)},$$

whose square is

$$x^4 + 4x^3y + 10x^2y^2 + 12xy^3 + 9y^4 \text{...........(ii)},$$

both expressions being arranged according to descending powers of x.

We may write the square of $x^2 + 2xy + 3y^2$ in either of the following forms:

$$\{x^2 + (2xy + 3y^2)\}^2 = x^4 + 2x^2(2xy + 3y^2) + (2xy + 3y^2)^2 \text{...(iii)},$$

$$\{(x^2 + 2xy) + 3y^2\}^2 = (x^2 + 2xy)^2 + 2(x^2 + 2xy)\,3y^2 + (3y^2)^2 \text{...(iv)}.$$

Now it is clear that the first term of (ii) is the square of the first term of (i). Hence the *first term* of the root of (ii) is found by taking the square root of its first term.

Again, we see from (iii) that when we have subtracted x^4 (the square of the first term of the root) the term in the remainder which contains the highest power of x is $2x^2 \times 2xy$, which is twice the product of the first and second terms of the root.

Hence, after subtracting from (ii) the square of the first term of the root, the *second term* is obtained by *dividing the first term of the remainder by twice the first term of the root.*

Again, we see from (iv) that when we have subtracted $(x^2 + 2xy)^2$, that is the square of the part of the root already found, the term in the remainder which contains the highest power of x is $2x^2 \times 3y^2$, which is twice the product of the first and third terms of the root.

Hence, after subtracting from (ii) the square of that part of the root already found, the *next term* of the root is obtained by *dividing the first term of the remainder by twice the first term of the root.*

If we now subtract the square of $x^2 + 2xy + 3y^2$ from the given expression there will be no remainder; and hence $x^2 + 2xy + 3y^2$ is the required root.

We will now consider the most general case.

Suppose we have to find the square root of $(A + B)^2$, where A stands for any number of terms of the root, and B for the rest; the terms in A and B being arranged according to descending (or ascending) powers of some letter, so that every term in A is of higher (or lower) dimensions than any term in B.

Also suppose that the terms in A are known, and that we have to find the terms in B.

Subtracting A^2 from $(A + B)^2$, we have the remainder $(2A + B)B$.

Now from the mode of arrangement it follows that the term of the highest (or lowest) degree in the remainder is twice the product of the first term in A and the first term in B.

Hence, to obtain the *next term* of the required root, that is, to obtain the highest (or lowest) term of B, we *subtract from the whole expression the square of that part of the root which is already found, and divide the first term of the remainder by twice the first term of the root.*

The first term of the root is the square root of the first term of the given expression; and when we have found the first term of the root we can find each of the other terms in succession by the above process.

The process of finding the square root of an algebraical expression can be written as follows:

$$\begin{array}{rl} & \underline{x^4 + 4x^3y + 10x^2y^2 + 12xy^3 + 9y^4}\ (\,x^2 + 2xy + 3y^2 \\ (x^2)^2 = & \underline{x^4 \qquad\qquad\qquad} \\ (x^2 + 2xy)^2 = & \underline{x^4 + 4x^3y + 4x^2y^2} \\ (x^2 + 2xy + 3y^2)^2 = & \underline{x^4 + 4x^3y + 10x^2y^2 + 12xy^3 + 9y^4} \end{array}$$

We first take the square root of the first term of the given expression: we thus obtain x^2, the *first term* of the required root.

Now subtract the square of x^2 from the given expression, and divide the first term of the remainder by $2x^2$: we thus obtain $2xy$, the *second term* of the root.

Now subtract the square of $x^2 + 2xy$, which is

$$x^4 + 4x^3y + 4x^2y^2,$$

from the given expression, and divide the first term of the remainder, namely $6x^2y^2$, by $2x^2$: we thus obtain $3y^2$, the *third term* of the root.

Subtract the square of $x^2 + 2xy + 3y^2$ from the given expression and there is no remainder.

Hence $x^2 + 2xy + 3y^2$ is the required square root.

The squares of x^2, $x^2 + 2xy$, &c. are placed under the given expression, like terms being placed in the same column: the remainder left after taking away any square is then obvious.

NOTE. Instead of finding each square independently, some labour can be saved by making use of the previous square. Thus, in the example just considered, the square of $x^2 + 2xy + 3y^2$ can be found by adding

$$2(x^2 + 2xy)(3y^2) + (3y^2)^2$$

to the preceding square $(x^2 + 2xy)^2$.

Beginners are however recommended to form the squares independently, as facility in *writing down* the squares of expressions is very useful, and the saving of labour is very slight except when there are many terms in the root.

The method of finding the cube root of any algebraical expression will be given in a subsequent chapter.

EXAMPLES. XLIII.

Write down the square roots of the following expressions:—

1. $9x^2 - 30xy^2 + 25y^4$. 2. $4x^{10} - 12x^5y^3 + 9y^6$.
3. $x^8 - 6x^4y^6 + 9y^{12}$. 4. $\frac{1}{4}x^8 - \frac{1}{3}x^4y^5 + \frac{1}{9}y^{10}$.
5. $a^2 + 4b^2 + 9c^2 + 12bc + 6ca + 4ab$.
6. $4a^2 + b^2 + 9c^2 + 6bc - 12ca - 4ab$.
7. $4a^4 + b^4 + c^4 - 2b^2c^2 - 4c^2a^2 + 4a^2b^2$.
8. $25a^4 + 9b^4 + 4c^4 - 12b^2c^2 + 20c^2a^2 - 30a^2b^2$.

Find the square roots of the following expressions:—

9. $x^4 + 2x^3 + 3x^2 + 2x + 1$. 10. $4x^4 - 8x^3 + 4x + 1$.
11. $9x^4 - 36x^3 + 72x + 36$. 12. $1 - xy - \frac{15}{4}x^2y^2 + 2x^3y^3 + 4x^4y^4$.
13. $4x^4 + 4x^3 - \frac{1}{2}x + \frac{1}{16}$. 14. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.
15. $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$.
16. $16 - 96x + 216x^2 - 216x^3 + 81x^4$.
17. $1 + 4x + 10x^2 + 12x^3 + 9x^4$. 18. $4x^4 - 4x^3 + 3x^2 - x + \frac{1}{4}$.
19. $(1 + 2x^2)^2 - 4x(1 - x)(1 + 2x)$.
20. $x^6 - 4x^5 + 6x^4 - 8x^3 + 9x^2 - 4x + 4$.
21. $9x^6 - 12x^5 + 22x^4 + x^2 + 12x + 4$.
22. $x^6 - 22x^4 + 34x^3 + 121x^2 - 374x + 289$.
23. $a^2 - ax + \frac{1}{4}x^2 + 8a - 4x + 16$.
24. $x^4 + 2x^3(y + z) + x^2(y^2 + z^2 + 4yz) + 2xyz(y + z) + y^2z^2$.

CHAPTER XIX.

INDICES.

171. WE have hitherto supposed that an *index* was always a positive integer; and this is necessarily the case so long as we retain the definition of a^n given in Art. 10; for, with that definition, such an expression as $a^{\frac{3}{5}}$, for example, has no meaning whatever.

It is however found convenient to extend the meaning of a^n, so as to include fractional and negative values of n.

172. Now it is very important that algebraical symbols should always obey the same laws; and to secure this result in the case of the symbol a^n, we do not assign any particular meanings to a^n, when n is negative or fractional, but first impose the restriction that *the meaning of* a^n *must be such that the fundamental index law*, namely

$$a^m \times a^n = a^{m+n},$$

shall always be true; and we shall find that the above restriction is of itself sufficient to define the meaning of a^n in all cases.

For example, to find the meaning of $a^{\frac{1}{2}}$.

Since the meaning is to be consistent with the Index Law, we must have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

Thus $a^{\frac{1}{2}}$ must be such that its square is a, that is

$$a^{\frac{1}{2}} = \sqrt{a}.$$

Again, to find the meaning of a^{-1}.

By the index law

$$a^{-1} \times a^2 = a^{-1+2} = a^1 = a;$$

$$\therefore \quad a^{-1} = a \div a^2 = \frac{1}{a}.$$

Thus $$a^{-1} = \frac{1}{a}.$$

We now proceed to consider the most general cases.

173. To find the meaning of $a^{\frac{1}{n}}$, where n is any positive integer.

By the index law

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{to } n \text{ factors}$$

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\text{to } n \text{ terms}} = a^1 = a.$$

Hence $a^{\frac{1}{n}}$ must be such that its nth power is a.

Thus $a^{\frac{1}{n}}$ *is the* n*th root of* a.

174. To find the meaning of $a^{\frac{m}{n}}$, where m and n are any positive integers.

By the index law

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots\text{to } n \text{ terms}} = a^m.$$

Thus $a^{\frac{m}{n}}$ *is equal to the* n*th root of* a^m.

We have also

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{ to } m \text{ factors}$$

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\text{ to } m \text{ terms}} = a^{\frac{m}{n}}.$$

Hence $a^{\frac{m}{n}}$ may be considered as the mth power of $a^{\frac{1}{n}}$, and by Art. 173,

$$a^{\frac{1}{n}} \text{ is } \sqrt[n]{a}.$$

Thus we may consider that $a^{\frac{m}{n}}$ is the nth root of the mth power of a, or that it is the mth power of the nth root of a; which we express by

$$a^{\frac{m}{n}} = \sqrt[n]{(a^m)} = (\sqrt[n]{a})^m.$$

175. To find the meaning of a^0.

By the index law

$$a^0 \times a^m = a^{0+m} = a^m;$$

$$\therefore \; a^0 = a^m \div a^m = 1.$$

Thus $a^0 = 1$, whatever a may be.

176. To find the meaning of a^{-m}, where m has any positive value.

By the index law

$$a^m \times a^{-m} = a^{m-m} = a^0,$$

and, by the preceding Article, $a^0 = 1$;

$$\therefore \quad a^m \times a^{-m} = 1,$$

so that

$$a^{-m} = \frac{1}{a^m}, \text{ and } a^m = \frac{1}{a^{-m}}.$$

Thus we can change any quantity from the numerator to the denominator, or from the denominator to the numerator, of a fraction, provided we change the sign of its index.

For example

$$\frac{a^3b^2}{x^3y} = a^3b^2x^{-3}y^{-1}$$

$$= \frac{1}{a^{-3}b^{-2}x^3y}.$$

177. To shew that $(a^m)^n = a^{mn}$, for all values of m and n.

I. If n be a positive integer, m having any value whatever, we have

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots \text{ to } n \text{ factors}\\ &= a^{m+m+m+\ldots \text{ to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

II. If n be a positive fraction $\frac{p}{q}$, where p and q are positive integers, we have

$$\begin{aligned}(a^m)^n = (a^m)^{\frac{p}{q}} &= \sqrt[q]{\{(a^m)^p\}} && \text{[Art. 174.]}\\ &= \sqrt[q]{\{a^{mp}\}} && \text{by I.}\\ &= a^{\frac{mp}{q}} && \text{[Art. 174.]}\\ &= a^{mn}.\end{aligned}$$

III. If n be negative, and equal to $-p$, we have

$$\begin{aligned}(a^m)^n = (a^m)^{-p} &= \frac{1}{(a^m)^p} && \text{[Art. 176.]}\\ &= \frac{1}{a^{mp}} && \text{by I. and II.}\\ &= a^{-mp} && \text{[Art. 176.]}\\ &= a^{m(-p)} = a^{mn}.\end{aligned}$$

Hence $(a^m)^n = a^{mn}$, for *all values* of m and n.

178. We now give some examples involving fractional and negative indices.

$$16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{16}} = \frac{1}{2}.$$

$$27^{\frac{4}{3}} = (\sqrt[3]{27})^4 = 3^4 = 81.$$

$$a^{\frac{2}{3}} \times a^{-\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}.$$

$$a^{\frac{1}{2}} \times a^{\frac{1}{6}} \times a^{-\frac{2}{3}} = a^{\frac{1}{2}+\frac{1}{6}-\frac{2}{3}} = a^0 = 1.$$

$$(a^{-\frac{2}{3}})^6 = a^{-\frac{2}{3}\times 6} = a^{-4} = \frac{1}{a^4}.$$

$$a^{-2} \div a^{-3} = a^{-2-(-3)} = a^{-2+3} = a.$$

$$\sqrt{(a^5 b^{-3} c^{-4})} = \sqrt{(a^5)} . \sqrt{(b^{-3})} . \sqrt{(c^{-4})} = a^{\frac{5}{2}} b^{-\frac{3}{2}} c^{-2}.$$

Multiply

$$a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}} \text{ by } a^{\frac{1}{3}} - 1 + a^{-\frac{1}{3}}.$$

$$\begin{array}{lllll}
a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}} & & & & \\
a^{\frac{1}{3}} - 1 + a^{-\frac{1}{3}} & & & & \\
\hline
a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1 & & & & \\
\quad - a^{\frac{1}{3}} - 1 - a^{-\frac{1}{3}} & & & & \\
\qquad\quad + 1 + a^{-\frac{1}{3}} + a^{-\frac{2}{3}} & & & & \\
\hline
a^{\frac{2}{3}} + 1 + a^{-\frac{2}{3}} & & & &
\end{array}$$

EXAMPLES. XLIV.

Find the values of

1. $8^{\frac{2}{3}}$.
2. $4^{-\frac{3}{2}}$.
3. $16^{-\frac{1}{2}}$.
4. $(\frac{1}{25})^{\frac{1}{2}}$.
5. $(\frac{1}{10000})^{-\frac{3}{4}}$.
6. $(\frac{1}{125})^{-\frac{2}{3}}$.

Simplify:

7. $a^{\frac{2}{3}}b^{\frac{4}{5}} \times a^{\frac{4}{3}}b^{\frac{6}{5}}$.
8. $\{a^{-\frac{2}{3}}b^{\frac{1}{6}}\}^{\frac{3}{2}}$.
9. $(a^{-\frac{1}{2}}b^{-2})^{-4}$.
10. $(a^4 b^{\frac{2}{3}})^{-\frac{3}{2}}$.

Express with fractional indices:

11. $\sqrt[3]{a} + \sqrt{x} + \sqrt[4]{y^3}$.
12. $\sqrt{a^3 y} + \sqrt[3]{a y^3}$.
13. $\sqrt[3]{a^2 x^5} + \sqrt{a^3 y}$.
14. $\sqrt[5]{(ab^2c^5)} \div \sqrt[3]{a^2 c^3}$.

Express the following quantities without the use of fractional or negative indices:

15. $a^{\frac{1}{2}} - a^{-2}$.

16. $a^{-4}b^{-\frac{1}{4}}$.

17. $a^{\frac{2}{3}}b^{-1} - a^{-\frac{2}{3}}b$.

18. $a^{-3}b^{-\frac{1}{2}} + 3a^{\frac{2}{3}}b^{-\frac{2}{3}}$.

Multiply

19. $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

20. $1 + x^{\frac{1}{4}} + x^{\frac{1}{2}}$ by $1 - x^{\frac{1}{4}}$.

21. $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

22. $x^{\frac{5}{4}} - x^{\frac{3}{4}} + x^{\frac{1}{4}} - x^{-\frac{1}{4}}$ by $x^{\frac{3}{4}} + x^{\frac{1}{4}}$.

23. $x^n + x^{\frac{n}{2}} + 1$ by $x^{-n} + x^{-\frac{n}{2}} + 1$.

24. $\frac{1}{8}a^{\frac{3}{2}} - \frac{1}{12}ab^{\frac{1}{2}} + \frac{1}{18}a^{\frac{1}{2}}b - \frac{1}{27}b^{\frac{3}{2}}$ by $\frac{1}{2}a^{\frac{1}{2}} - \frac{1}{3}b^{\frac{1}{2}}$.

Divide

25. $x^3 - y^3$ by $x^{\frac{3}{2}} - y^{\frac{3}{2}}$.

26. $x^{\frac{3n}{2}} - y^{\frac{3n}{2}}$ by $x^{\frac{n}{2}} - y^{\frac{n}{2}}$.

27. $x^2 + y^2$ by $x^{\frac{2}{3}} + y^{\frac{2}{3}}$.

28. $x + 243y^{\frac{5}{3}}$ by $x^{\frac{1}{5}} + 3y^{\frac{1}{3}}$.

29. Simplify $(ab^{-3}c^5)^{\frac{1}{2}} \times (a^7b^4c^{-1})^{\frac{1}{3}} \times (a^{-5}bc^2)^{\frac{1}{6}}$.

30. Simplify $(a - 4b^{\frac{2}{3}})(a + 2a^{\frac{1}{2}}b^{\frac{1}{3}} + 4b^{\frac{2}{3}})(a - 2a^{\frac{1}{2}}b^{\frac{1}{3}} + 4b^{\frac{2}{3}})$.

31. Shew that $1 - \dfrac{a^2 + b^2 - a^{-2} - b^{-2}}{a^2b^2 - a^{-2}b^{-2}} = \dfrac{(a - a^{-1})(b - b^{-1})}{ab + a^{-1}b^{-1}}$.

32. Write down the square root

(i) of $x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 1$, (ii) of $4x^{\frac{2}{3}} - 4x^{\frac{1}{3}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$,

and (iii) of $ax^{\frac{2}{3}} - 2a^{\frac{5}{6}}x^{\frac{5}{6}} + a^{\frac{2}{3}}x$.

33. Find the square root of $4x^2a^{-2} - 12xa^{-1} + 25 - 24x^{-1}a + 16x^{-2}a^2$.

34. Find the square root of $x^{\frac{2}{3}} - 4x^{\frac{5}{6}} + 4x + 2x^{\frac{7}{6}} - 4x^{\frac{4}{3}} + x^{\frac{5}{3}}$.

CHAPTER XX.

SURDS.

179. WE shall in the present chapter consider properties of *surds*, making use of the principles established in the two preceding chapters.

180. Any rational quantity can be written in the form of a surd.

For example,

$$a = \sqrt[2]{a^2} = \sqrt[3]{a^3} = \sqrt[n]{a^n},$$

and $$2 = \sqrt[2]{4} = \sqrt[3]{8} = \sqrt[n]{2^n}.$$

Also, since $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, we have

$$3\sqrt{2} = \sqrt{3^2} \times \sqrt{2} = \sqrt{(3^2 \times 2)} = \sqrt{18},$$

$$2\sqrt[3]{5} = \sqrt[3]{2^3} \times \sqrt[3]{5} = \sqrt[3]{(2^3 \times 5)} = \sqrt[3]{40},$$

and $$a\sqrt[n]{b} = \sqrt[n]{a^n} \times \sqrt[n]{b} = \sqrt[n]{(a^n b)}.$$

181. Surds can often be simplified by being written as the product of a rational and an irrational factor.

For since

$$\sqrt[n]{(ab)} = \sqrt[n]{a} \times \sqrt[n]{b},$$

we have

$$\sqrt{12} = \sqrt{(4 \times 3)} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3},$$

$$\sqrt[3]{54} = \sqrt[3]{(27 \times 2)} = \sqrt[3]{27} \times \sqrt[3]{2} = 3\sqrt[3]{2},$$

$$\sqrt{75} - \sqrt{27} = \sqrt{(5^2 \times 3)} - \sqrt{(3^2 \times 3)}$$

$$= 5\sqrt{3} - 3\sqrt{3} = 2\sqrt{3},$$

and $$\sqrt{a^3} + \sqrt{ab^2} = a\sqrt{a} + b\sqrt{a} = (a+b)\sqrt{a}.$$

182. Surds are said to be of the same *order* when the same root is required to be taken. Thus $\sqrt[3]{3}$, $\sqrt[3]{(a+x)}$ and $5^{\frac{2}{3}}$ are all surds of the *third* order.

A surd of the second order, for example $\sqrt{2}$, is often called a *quadratic* surd.

183. Any two surds can be reduced to surds of the same order.

Let $a^{\frac{m}{n}}$ and $b^{\frac{p}{q}}$ be the two surds; then we have to change these into equivalent surds which are such that their indices have the same denominator.

We can reduce both surds to the order nq, as follows:

$$a^{\frac{m}{n}} = a^{\frac{mq}{nq}} = \sqrt[nq]{a^{mq}},$$

and $$b^{\frac{p}{q}} = b^{\frac{pn}{qn}} = \sqrt[nq]{b^{pn}}.$$

Thus the required surds are $\sqrt[nq]{a^{mq}}$ and $\sqrt[nq]{b^{pn}}$.

Ex. 1. Reduce $\sqrt{2}$ and $\sqrt[3]{3}$ to surds of the same order.

The L.C.M. of the denominators of the indices of the surds $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ is 6; and the given surds can both be reduced to the sixth order. We have

$$2^{\frac{1}{2}} = 2^{\frac{3}{6}} = (2^3)^{\frac{1}{6}} = \sqrt[6]{8},$$

$$3^{\frac{1}{3}} = 3^{\frac{2}{6}} = (3^2)^{\frac{1}{6}} = \sqrt[6]{9}.$$

Thus the required surds are $\sqrt[6]{8}$ and $\sqrt[6]{9}$.

Ex. 2. Reduce $\sqrt[4]{3}$ and $\sqrt[6]{5}$ to surds of the same order.

The L.C.M. of the denominators of the indices of the surds $3^{\frac{1}{4}}$ and $5^{\frac{1}{6}}$ is 12.

Hence we have

$$3^{\frac{1}{4}} = 3^{\frac{3}{12}} = (3^3)^{\frac{1}{12}} = \sqrt[12]{27},$$

$$5^{\frac{1}{6}} = 5^{\frac{2}{12}} = (5^2)^{\frac{1}{12}} = \sqrt[12]{25}.$$

Thus $\sqrt[12]{27}$ and $\sqrt[12]{25}$ are the required surds.

Ex. 3. Which is the greater $\sqrt[3]{14}$ or $\sqrt[2]{6}$?

We must reduce the surds to equivalent surds of the same order. Thus

$$14^{\frac{1}{3}} = 14^{\frac{2}{6}} = (14^2)^{\frac{1}{6}} = \sqrt[6]{196},$$

and

$$6^{\frac{1}{2}} = 6^{\frac{3}{6}} = (6^3)^{\frac{1}{6}} = \sqrt[6]{216}.$$

But $\sqrt[6]{216}$ is greater than $\sqrt[6]{196}$;

$\therefore$ $\sqrt[2]{6}$ is greater than $\sqrt[3]{14}$.

184. The product of any two surds of the same order can be written down at once from the formula

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{(ab)}.$$

When surds are of different orders, they must first be reduced to equivalent surds of the same order: their product can then be found as above.

For example,

$$\sqrt[3]{5} \times \sqrt[3]{20} = \sqrt[3]{(5 \times 20)} = \sqrt[3]{100},$$

and $$\sqrt{2} \times \sqrt[3]{3} = \sqrt[6]{8} \times \sqrt[6]{9} = \sqrt[6]{(8 \times 9)} = \sqrt[6]{72}.$$

Also $$(\sqrt{2} + \sqrt{3})(\sqrt{2} + 2\sqrt{5})$$
$$= \sqrt{2}\sqrt{2} + \sqrt{3}\sqrt{2} + 2\sqrt{2}\sqrt{5} + 2\sqrt{3}\sqrt{5}$$
$$= 2 + \sqrt{6} + 2\sqrt{10} + 2\sqrt{15}.$$

185. When surds occur in the denominators of fractions they can be got rid of, and the denominators are then said to have been rationalised.

The following examples will sufficiently illustrate the process:

$$\frac{3\sqrt{2}}{\sqrt{7}} = \frac{3\sqrt{2}\sqrt{7}}{\sqrt{7}\sqrt{7}} = \frac{3}{7}\sqrt{14},$$

$$\frac{2+\sqrt{5}}{\sqrt{5}-1} = \frac{(2+\sqrt{5})(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{2\sqrt{5}+5+2+\sqrt{5}}{5-1}$$
$$= \tfrac{1}{4}(7+3\sqrt{5}),$$

$$\frac{a-\sqrt{b}}{a+\sqrt{b}} = \frac{(a-\sqrt{b})(a-\sqrt{b})}{(a+\sqrt{b})(a-\sqrt{b})} = \frac{a^2-2a\sqrt{b}+b}{a^2-b}.$$

It is important to notice that $a \pm \sqrt{b}$ is made rational by multiplying by $a \mp \sqrt{b}$; also that $\sqrt{a} \pm \sqrt{b}$ is made rational by multiplying by $\sqrt{a} \mp \sqrt{b}$.

When the denominator of a fraction is rationalised its numerical value can be more easily found.

186. The following is an important proposition.

If $a + \sqrt{b} = \alpha + \sqrt{\beta}$, where a and α are rational, and $\sqrt{b}$ and $\sqrt{\beta}$ are irrational; then will $a = \alpha$, and $b = \beta$.

For we have $$a - \alpha + \sqrt{b} = \sqrt{\beta}.$$
Square both sides; then, after transformation, we have
$$2(a-\alpha)\sqrt{b} = \beta - b - (a-\alpha)^2.$$

Hence, *unless the coefficient of* $\sqrt{b}$ *is zero*, we must have an irrational and a rational quantity equal to one another, and this is impossible.

The coefficient of $\sqrt{b}$ in the last equation must therefore be zero; hence $a = \alpha$. And when $a = \alpha$, we have from the given relation, $\sqrt{b} = \sqrt{\beta}$, or $b = \beta$.

Hence, if the sum (or difference) of a rational quantity and a quadratic surd be equal to the sum (or difference) of another rational quantity and a quadratic surd, the two rational quantities must be equal to one another, as also the two irrational quantities.

It should be noticed that when $a + \sqrt{b} = \alpha + \sqrt{\beta}$, we can only conclude that $a = \alpha$ and $b = \beta$ provided that $\sqrt{b}$ and $\sqrt{\beta}$ are really irrational. We cannot, for example, from the relation $3 + \sqrt{4} = 2 + \sqrt{9}$, conclude that $3 = 2$ and $4 = 9$.

187. The square root of a binomial expression which is the sum of a rational quantity and a quadratic surd can sometimes be found in a simple form. The process is as under.

To find $\sqrt{(a + \sqrt{b})}$, where $\sqrt{b}$ is a surd.

Let $$\sqrt{(a + \sqrt{b})} = \sqrt{\alpha} + \sqrt{\beta} \quad\text{..................(i).}$$

Square both sides; then

$$a + \sqrt{b} = \alpha + \beta + 2\sqrt{\alpha\beta}.$$

Now, since $\sqrt{b}$ is a surd, we can equate the rational and irrational parts; hence

$$\left.\begin{aligned} a &= \alpha + \beta \\ b &= 4\alpha\beta \end{aligned}\right\} \quad\text{.......... ....(ii).}$$

Hence α and β are the roots of the equation

$$x^2 - ax + \frac{b}{4} = 0,$$

and these roots are

$$\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\} \text{ and } \left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}.$$

Thus $$\sqrt{(a+\sqrt{b})}=\sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}}$$
$$+\sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

It is clear, that unless $\sqrt{(a^2-b)}$ is rational, the right side of (iii) is much more complicated than the left. Thus the above process fails entirely unless a^2-b is a square number; and as this condition will not often be satisfied, the process has no great practical utility.

From (ii) we see that we have to find two numbers whose sum is a and whose product is $\frac{b}{4}$; and if two *rational* numbers satisfy these conditions, they can generally be found at once by inspection.

Ex. 1. Find $\sqrt{(15+2\sqrt{56})}$.

Let $$\sqrt{(15+2\sqrt{56})}=\sqrt{\alpha}+\sqrt{\beta}.$$

Square both sides; then

$$15+2\sqrt{56}=\alpha+\beta+2\sqrt{\alpha\beta}.$$

Equalizing the rational and irrational terms, we have

$$\alpha+\beta=15,$$
$$\alpha\beta=56.$$

The numbers which satisfy these relations are obviously 7 and 8. Hence

$$\sqrt{(15+2\sqrt{56})}=\sqrt{7}+\sqrt{8}.$$

Ex. 2. Find $\sqrt{(6-\sqrt{35})}$.

Let $\sqrt{(6-\sqrt{35})}=\sqrt{\alpha}-\sqrt{\beta}$.

Square both sides; then

$$6-\sqrt{35}=\alpha+\beta-2\sqrt{\alpha\beta}.$$

Hence, equating the rational and irrational terms, we have

$$\alpha+\beta=6,$$
$$\alpha\beta=\tfrac{35}{4}.$$

Hence $\alpha=\tfrac{7}{2},\ \beta=\tfrac{5}{2}.$

Thus $\sqrt{(6-\sqrt{35})}=\sqrt{\tfrac{7}{2}}-\sqrt{\tfrac{5}{2}}.$

EXAMPLES. XLV.

Simplify

1. $\sqrt{27}+\sqrt{48}$.
2. $3\sqrt[3]{5}+\sqrt[3]{625}$.
3. $2\sqrt{180}-\sqrt{405}$.
4. $\sqrt{512}-\sqrt{50}-\sqrt{98}$.
5. $\sqrt{147}-2\sqrt{27}-\sqrt{3}$.
6. $\sqrt{15}\times\sqrt{60}$.
7. $\sqrt[3]{6}\times\sqrt[3]{36}$.
8. $2\sqrt{\dfrac{2}{3}}\times3\sqrt{\dfrac{3}{2}}$.
9. $\sqrt{10}\times\sqrt[3]{200}$.
10. $\sqrt[3]{4}\times\sqrt[2]{8}$.

Rationalize the denominators of the following fractions:

11. $\dfrac{\sqrt{3}}{\sqrt{5}}$.
12. $\sqrt{\dfrac{2}{3}}$.
13. $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.
14. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.
15. $\dfrac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$.
16. $\dfrac{15+14\sqrt{3}}{15-2\sqrt{3}}$.
17. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$.
18. $\dfrac{3}{2+\sqrt{3}+\sqrt{5}}$.
19. $\dfrac{1}{\sqrt[3]{2}-1}+\dfrac{3}{\sqrt[3]{2}+1}$.
20. $\dfrac{4}{\sqrt[3]{9}-1}+\dfrac{5}{\sqrt[3]{9}+1}$.

21. Find the value of $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ to three places of decimals.

22. Find the value of $\frac{\sqrt{3}}{2-\sqrt{3}}+\frac{2-\sqrt{2}}{2+\sqrt{2}}$ to three places of decimals.

Find the square roots of

23. $6+\sqrt{20}$.

24. $16+6\sqrt{7}$.

25. $12-6\sqrt{3}$.

26. $28-5\sqrt{12}$.

27. $101-28\sqrt{13}$.

28. $117-36\sqrt{10}$.

29. $280+56\sqrt{21}$.

30. $4\frac{1}{2}+2\sqrt{2}$.

31. $3\sqrt{5}-\sqrt{2}+\sqrt{(7+2\sqrt{10})}$.

32. $6-4\sqrt{3}+\sqrt{(16-8\sqrt{3})}$.

33. $2a+2\sqrt{(a^2-x^2)}$.

34. $3x-1+2\sqrt{(2x^2+x-6)}$.

Simplify

35. $\frac{1}{\sqrt{(16+6\sqrt{7})}}$.

36. $\frac{1}{\sqrt{(15+2\sqrt{56})}}$.

37. $\sqrt{(6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6})}$.

CHAPTER XXI.

RATIO. PROPORTION. VARIATION.

188. THE relative magnitude of two quantities, measured by the number of times which the one contains the other, is called their *ratio.*

Concrete quantities of different kinds can have no ratio to one another: we cannot, for example, compare miles with tons, or shillings with weeks.

The ratio of a to b is expressed by the notation $a : b$; and a is called the first *term*, and b the second *term* of the ratio.

Sometimes the first and second terms of a ratio are called respectively the *antecedent* and the *consequent.*

189. Magnitudes must always be expressed by means of *numbers;* and the number of times which one number is contained in another is found by dividing the one by the other. Hence

$$a : b \text{ is equal to } \frac{a}{b}.$$

Thus ratios can be expressed as fractions.

190. A fraction is unaltered in value by multiplying its numerator and denominator by the same number.

Hence also a ratio is unaltered in value by multiplying each of its terms by the same number. Thus the ratios

$$2 : 3, \quad 6 : 9 \quad \text{and} \quad 2m : 3m$$

are all equal to one another.

Again, the ratios $4 : 5$, $7 : 9$ and $11 : 15$ are equal respectively to

$$36 : 45, \quad 35 : 45 \text{ and } 33 : 45.$$

Hence the ratios $4 : 5$, $7 : 9$ and $11 : 15$ are in descending order of magnitude.

191. It is clear that a ratio is greater, equal, or less than unity, according as its first term is greater, equal, or less than the second.

A ratio which is greater than unity is sometimes called a ratio of *greater inequality*, and a ratio which is less than unity is similarly called a ratio of *less inequality*.

192. A ratio is altered in value when the same quantity is *added* to each of its terms.

For example, by adding 1, 10 and 100 to each of the terms of the ratio $4 : 5$, we obtain respectively the ratios

$$5 : 6, \quad 14 : 15 \quad \text{and} \quad 104 : 105;$$

and these new ratios are different from the given ratio and from each other.

Since

$$\tfrac{4}{5} = 1 - \tfrac{1}{5}, \quad \tfrac{14}{15} = 1 - \tfrac{1}{15} \quad \text{and} \quad \tfrac{104}{105} = 1 - \tfrac{1}{105},$$

we see that by adding the same quantity to each of the terms of the ratio $4 : 5$, a new ratio is obtained which becomes more nearly equal to unity as the quantity added becomes greater.

This is a particular case of the following general proposition.

193. *Any ratio is made more nearly equal to unity by adding the same quantity to each of its terms.*

By adding x to each term of the ratio $a : b$, the ratio $a+x : b+x$ is obtained. We have to shew that $\frac{a+x}{b+x}$ is more nearly equal to 1 than is $\frac{a}{b}$.

Now $$\frac{a}{b}-1=\frac{a-b}{b},$$

also $$\frac{a+x}{b+x}-1=\frac{a-b}{b+x},$$

and it is clear that the *absolute value* of $\frac{a-b}{b+x}$ is less than that of $\frac{a-b}{b}$: this proves the proposition.

When a is greater than b, $\frac{a}{b}$ is greater than 1, and so also is $\frac{a+x}{b+x}$; but $\frac{a+x}{b+x}$ is more nearly equal to unity than $\frac{a}{b}$ is, and therefore $\frac{a+x}{b+x}$ is less than $\frac{a}{b}$.

Thus a ratio which is greater than unity is *diminished* by adding the same quantity to each of its terms.

If however a is less than b, $\frac{a}{b}$ is less than 1, and so also is $\frac{a+x}{b+x}$; but $\frac{a+x}{b+x}$ is more nearly equal to unity than $\frac{a}{b}$ is, and therefore $\frac{a+x}{b+x}$ is greater than $\frac{a}{b}$.

Thus a ratio which is less than unity is *increased* by adding the same quantity to each of its terms.

Also, when x is very great, the fraction $\frac{a-b}{b+x}$ is very small; and we can make $\frac{a-b}{b+x}$, which is the difference between $\frac{a+x}{b+x}$ and 1, *as small as we please* by taking x sufficiently great. This is expressed by saying that the *limiting value* of $\frac{a+x}{b+x}$, when x is very great, is unity.

194. The following definitions are sometimes required:

The ratio of the product of the first terms of any number of ratios to the product of their second terms is called the ratio *compounded* of the given ratios.

Thus $ac : bd$ is called the ratio compounded of the ratios

$$a : b \text{ and } c : d.$$

The ratio $a^2 : b^2$ is called the *duplicate ratio* of $a : b$.

The ratio $a^3 : b^3$ is called the *triplicate ratio* of $a : b$.

The ratio $\sqrt{a} : \sqrt{b}$ is called the *sub-duplicate* ratio of $a : b$.

195. The ratio of two quantities cannot always be expressed by the ratio of two whole numbers; for example, the ratio of a diagonal to a side of a square cannot be so expressed, for this ratio is $\sqrt{2}$, and we cannot find any fraction which is *exactly* equal to $\sqrt{2}$.

When the ratio of two quantities cannot be exactly expressed by the ratio of two whole numbers, they are said to be *incommensurable quantities.*

EXAMPLES. XLVI.

1. Arrange the ratios $5 : 6$, $7 : 8$, $41 : 48$, and $31 : 36$ in order of magnitude.

2. For what value of x will the ratio $3+x : 4+x$ be equal to $5 : 6$?

3. For what value of x will the ratio $15+x : 17+x$ be equal to $\frac{1}{2}$?

4. What must be added to each of the terms of $3 : 4$ to make the ratio equal to $25 : 32$?

5. Find two numbers in the ratio of 5 to 6, and whose sum is 121.

6. Two numbers are in the ratio of 3 to 8, and the sum of their squares is 3577: find them.

7. What is the ratio of x to y, if $\dfrac{4x+5y}{3x-y} = 2$?

8. If $4x^2 + y^2 = 4xy$, find the ratio of x to y.

9. Find $x : y$, having given $x^2 + 6y^2 = 5xy$.

10. A certain ratio will be equal to $2 : 3$ if 2 be added to each of its terms, and it will be equal to $1 : 2$ if 1 be subtracted from each of its terms: find the ratio.

11. Find two numbers such that their sum, their difference, and the sum of their squares are as $7 : 1 : 75$.

12. What is the least integer which must be added to the terms of the ratio $9 : 23$ so as to make it greater than the ratio $7 : 11$?

13. Write down the ratio compounded of the ratios $2 : 3$ and $15 : 16$; also the ratio compounded of the ratios $5 : 6$ and $18 : 25$.

14. Write down two quantities which are in the duplicate ratio of $2x : 3y$.

15. Find x in order that $x+1 : x+4$ may be the duplicate ratio of $3 : 5$.

PROPORTION.

196. Four quantities are said to be *proportional* when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus a, b, c, d are proportional, if

$$a : b = c : d.$$

This is sometimes expressed by the notation

$$a : b :: c : d,$$

which is read "a is to b as c is to d."

The first and fourth, of four quantities in proportion, are sometimes called the *extremes*, and the second and third of the quantities are called the *means*.

197. If the four quantities a, b, c, d are proportional, we have by definition

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply each of these equals by bd; then

$$ad = bc.$$

Thus *the product of the extremes is equal to the product of the means.*

Conversely, *if* $ad = bc$, *then* a, b, c, d *will be proportional.*

For, if $$ad = bc,$$

then $$\frac{ad}{bd} = \frac{bc}{bd};$$

$$\therefore \quad \frac{a}{b} = \frac{c}{d},$$

that is $$a : b = c : d.$$

Thus, if $a : b = c : d$, then $ad = bc$;

and if $ad = bc$, then $a : b = c : d$.

198. It follows from the last article that the four relations

$$a : b = c : d,$$

$$a : c = b : d,$$

$$b : a = d : c,$$

and $$b : d = a : c,$$

are all true, provided that $ad = bc$.

Hence the above four proportions are all true, when any one of them is true.

199. If $$a : b = c : d,$$

then will $$a + b : a - b = c + d : c - d.$$

For $$a + b : a - b = c + d : c - d,$$

if $$(a + b)(c - d) = (a - b)(c + d),$$

that is, if $$ac + bc - ad - bd = ac - bc + ad - bd,$$

or, if $$bc = ad.$$

But this condition is satisfied, since

$$a : b = c : d.$$

The above proposition may also be proved as follows:

Since $$a : b = c : d;$$

$$\therefore \quad \frac{a}{b} = \frac{c}{d};$$

$$\therefore \quad \frac{a}{b} + 1 = \frac{c}{d} + 1,$$

that is $$\frac{a + b}{b} = \frac{c + d}{d} \quad \text{........................(i).}$$

Also $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

that is $$\frac{a - b}{b} = \frac{c - d}{d} \quad \text{........................(ii).}$$

From (i) and (ii), by division,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d};$$

$$\therefore \quad a+b : a-b = c+d : c-d.$$

200. Quantities are said to be in *continued proportion* when the ratios of the first to the second, of the second to the third, of the third to the fourth, &c. are all equal.

Thus a, b, c, d, &c. are in continued proportion if

$$a : b = b : c = c : d, \text{ \&c.},$$

that is, if $$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\text{\&c.}$$

If $a : b = b : c$, then b is called the *mean proportional* between a and c; also c is called the *third proportional* to a and b.

201. If a, b, c be in continued proportion, we have

$$\frac{a}{b}=\frac{b}{c};$$

$$\therefore \quad b^2=ac, \text{ or } b=\sqrt{ac}.$$

Thus *the mean proportional between two given quantities is the square root of their product.*

Also $$\frac{a}{b}\times\frac{b}{c}=\frac{b}{c}\times\frac{b}{c},$$

that is $$\frac{a}{c}=\frac{b^2}{c^2}=\frac{a^2}{b^2}.$$

Hence $$a : c = a^2 : b^2.$$

Thus, *if three quantities are in continued proportion the ratio of the first to the third is the duplicate ratio of the first to the second.*

202. If $a : b = c : d = e : f$, then will each ratio be equal to $a + c + e : b + d + f$.

We have $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

and we have to shew that each fraction is equal to

$$\frac{a+c+e}{b+d+f}.$$

Let $\frac{a}{b} = x$; then will $\frac{c}{d} = x$, and $\frac{e}{f} = x$.

$$\therefore \quad a = bx, \; c = dx, \text{ and } e = fx.$$

Hence, by addition,

$$\begin{aligned} a + c + e &= bx + dx + fx \\ &= (b + d + f)\, x; \end{aligned}$$

$$\therefore \quad \frac{a+c+e}{b+d+f} = x.$$

This proves the proposition.

203. When $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are unequal, $\frac{a+c+e}{b+d+f}$ is greater than the least, and less than the greatest, of the three fractions $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$.

For, let $\frac{a}{b}$ be the greatest, and let

$$\frac{a}{b} = x.$$

Then $\frac{c}{d}$ is less than x, and also $\frac{e}{f}$ is less than x.

Hence $$a = xb,$$
$$c < xd,$$
and $$e < xf.$$

Hence $$a+c+e<x(b+d+f);$$

$$\therefore \quad \frac{a+c+e}{b+d+f}<x.$$

Thus $\frac{a+c+e}{b+d+f}$ is less than the greatest of the three fractions.

Similarly $\frac{a+c+e}{b+d+f}$ can be shewn to be greater than the least of the three fractions.

We can prove in a similar manner that

$$\frac{a+c+e+g+\ldots}{b+d+f+h+\ldots}$$

is greater than the least and less than the greatest of the fractions $$\frac{a}{b},\ \frac{c}{d},\ \frac{e}{f},\ \frac{g}{h},\ \ldots$$

however many fractions there may be.

204. It is often very convenient to represent a ratio by a single letter, as in Art. 202. The following are additional examples:

Ex. 1. If $$a : b = c : d,$$

then $$a^2+ab : c^2+cd = b^2-2ab : d^2-2cd.$$

Let $\frac{a}{b}=x$; then also $\frac{c}{d}=x$.

Hence $$a=bx, \text{ and } c=dx.$$

$$\therefore \quad \frac{a^2+ab}{c^2+cd}=\frac{b^2x^2+b^2x}{d^2x^2+d^2x}=\frac{b^2(x^2+x)}{d^2(x^2+x)}=\frac{b^2}{d^2}.$$

Also $$\frac{b^2-2ab}{d^2-2cd}=\frac{b^2-2b^2x}{d^2-2d^2x}=\frac{b^2(1-2x)}{d^2(1-2x)}=\frac{b^2}{d^2}.$$

Hence $$\frac{a^2+ab}{c^2+cd}=\frac{b^2-2ab}{d^2-2cd},$$

that is $$a^2+ab : c^2+cd = b^2-2ab : d^2-2cd.$$

Ex. 2. If $a : b = c : d = e : f$, shew that
$$a^3+c^3+e^3 : b^3+d^3+f^3 = ace : bdf.$$

Let $\frac{a}{b}=x$; then $\frac{c}{d}=x$ and $\frac{e}{f}=x$.

Hence $a=bx$, $c=dx$ and $e=fx$;

$$\therefore \quad \frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{b^3x^3+d^3x^3+f^3x^3}{b^3+d^3+f^3}=x^3.$$

And $$\frac{ace}{bdf}=\frac{bx \,.\, dx \,.\, fx}{bdf}=x^3.$$

Hence $$\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf},$$

that is $$a^3+c^3+e^3 : b^3+d^3+f^3 = ace : bdf.$$

205. The definition of proportion given in Euclid is as follows: Four quantities are proportionals, when if any equimultiples be taken of the first and the third, and also any equimultiples of the second and the fourth, the multiple of the third is always greater than, equal to, or less than the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than the multiple of the second.

If the four quantities a, b, c, d satisfy the algebraical test of proportionality, we have

$$\frac{a}{b}=\frac{c}{d};$$

therefore, for all values of m and n,

$$\frac{ma}{nb}=\frac{mc}{nd}.$$

Hence $$mc \gtreqless nd,$$

as $$ma \gtreqless nb.$$

Thus a, b, c, d satisfy also Euclid's test of proportionality.

Next, suppose that a, b, c, d satisfy Euclid's definition of proportion.

If a and b are commensurable, so that $a : b = m : n$, where m and n are whole numbers; then

$$\frac{a}{b} = \frac{m}{n};$$

$$\therefore \quad na = mb.$$

But, by definition,

$$nc \gtreqless md,$$

according as $$na \gtreqless mb.$$

Hence $$nc = md;$$

$$\therefore \quad \frac{c}{d} = \frac{m}{n} = \frac{a}{b}.$$

Thus a, b, c, d satisfy the algebraical definition.

If a and b are *incommensurable* we cannot find two *whole* numbers m and n such that $a : b = m : n$. But, if we take any multiple na of a, this must lie between two consecutive multiples, say mb and $(m+1)\,b$, of b, so that

$$na > mb \text{ and } na < (m+1)\,b.$$

Hence, by the definition,

$$nc > md \text{ and } nc < (m+1)\,d;$$

$$\therefore \quad \frac{c}{d} > \frac{m}{n} \text{ and } \frac{c}{d} < \frac{m+1}{n}.$$

Hence both $\frac{a}{b}$ and $\frac{c}{d}$ lie between

$$\frac{m}{n} \text{ and } \frac{m+1}{n}.$$

Thus the difference between $\frac{a}{b}$ and $\frac{c}{d}$ is less than $\frac{1}{n}$; and as this is the case *however great n may be*, $\frac{a}{b}$ must be equal to $\frac{c}{d}$.

EXAMPLES. XLVII.

1. Shew that, if $a : b :: c : d$, then

(i) $ac : bd :: c^2 : d^2$.

(ii) $ab : cd :: a^2 : c^2$.

(iii) $a^2 : c^2 :: a^2 - b^2 : c^2 - d^2$.

2. Shew that, if $a : b = c : d$, then

$$2a + 3c : 3a + 2c = 2b + 3d : 3b + 2d.$$

3. If $a : b :: a + c : b + d$, then

$$c : d :: c + a : d + b.$$

4. If $a : b = c : d$, then

$$la + mb : pa + qb = lc + md : pc + qd.$$

5. Shew that, if $3a - 5b : 3c - 5d = 5a + 3b : 5c + 3d$, then

$$a : b = c : d.$$

6. Find a mean proportional to a^3b and ab^3.

7. Find a mean proportional to $(a + b)^2$ and $(a - b)^2$.

8. Find a third proportional to a and a^2; also to

$$(a - b)^2 \text{ and } a^2 - b^2.$$

9. If $a : b :: c : d$, then will $ab + cd$ be a mean proportional between $a^2 + c^2$ and $b^2 + d^2$.

10. Shew that, if $a : b :: c : d$, then

(i) $a : a + c :: a + b : a + b + c + d$.

(ii) $a^2 + ab + b^2 : a^2 - ab + b^2 :: c^2 + cd + d^2 : c^2 - cd + d^2$.

(iii) $a+b : c+d :: \sqrt{a^2+b^2} : \sqrt{c^2+d^2}$.

(iv) $\sqrt{a^2+b^2} : \sqrt{c^2+d^2} :: \sqrt[3]{a^3+b^3} : \sqrt[3]{c^3+d^3}$.

(v) $a^2c+ac^2 : b^2d+bd^2 :: (a+c)^3 : (b+d)^3$.

(vi) $\sqrt[n]{a^n+b^n} : \sqrt[n]{c^n+d^n} :: \sqrt[r]{a^r-b^r} : \sqrt[r]{c^r-d^r}$.

11. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, then will $\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}$, and also

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2.$$

VARIATION.

206. When any substance is sold at a fixed price per pound, the cost of any amount of it is so related to its weight that when the weight is doubled the cost is also doubled, when the weight is halved the cost is also halved, and so on. Now two numbers are in the same ratio as their doubles, or as their halves, &c. Thus the number which represents the cost bears always the same ratio to the number which represents the weight. When two quantities are related in this way, the one is said to *vary as* the other.

We therefore have the following definition:

One quantity is said to *vary as* another when the two quantities are so related to one another that their measures are in a constant ratio.

The symbol $\propto$ is used for the words *varies as*. Thus $A \propto B$ is read 'A varies as B.'

207. If $A \infty B$, $A : B$ is constant by definition; hence, if we put m for the constant ratio, we have

$$\frac{A}{B}=m, \quad \text{or } A=mB.$$

To find the constant m in any case it is only necessary to know one set of corresponding values of A and B.

For example, if $A \propto B$, and A is 10 when B is 2, we have

$$A = mB;$$

$$\therefore \quad 10 = m \times 2,$$

or $$m = 5.$$

Hence $$A = 5B.$$

208. One quantity is said to *vary inversely as* another when the first varies as the reciprocal of the second. Thus A varies inversely as B, if $A : \frac{1}{B}$ is constant.

If $A : \frac{1}{B}$ is constant, and we put m for the constant ratio, we have

$$\frac{A}{\frac{1}{B}} = m, \quad \text{or} \quad A = \frac{m}{B}.$$

209. One quantity is said to vary as two others *jointly*, when the ratio of the first to the product of the two others is constant.

Thus A is said to vary as B and C *jointly*, if $A : BC$ is constant; that is, if $A = m \,.\, BC$, where m is a constant.

210. One quantity is said to *vary directly as a second and inversely as a third*, when the ratio of the first to the product of the second and the reciprocal of the third is constant.

Thus A is said to vary directly as B and inversely as C, if $A : B \times \frac{1}{C}$ is constant; that is, if $A = m\frac{B}{C}$, where m is a constant.

211. In all the different cases of variation defined above, the *constant* will be determined when any one set of corresponding values is given.

For example, if A varies as B and inversely as C, and A is 6 when B is 2 and C is 9, we have

$$A = m\frac{B}{C},$$

and $$6 = m \cdot \frac{2}{9};$$

$$\therefore \quad m = 27.$$

Hence $$A = 27\frac{B}{C}.$$

212. If $A \propto B$ when C is unchanged, and if $A \propto C$ when B is unchanged; then will $A \propto BC$ when B and C are both changed.

We may suppose the changes of B and C to be made successively.

Let then B be changed to b, and in consequence let A become a', C being unaltered.

Then, by supposition,

$$\frac{A}{B} = \frac{a'}{b} \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}.$$

Now let C be changed into c, and in consequence let a' become a, b being unaltered.

Then, by supposition,

$$\frac{a'}{C} = \frac{a}{c} \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}.$$

Hence, from (i) and (ii),

$$\frac{A}{BC} = \frac{a}{bc},$$

which proves the theorem.

The following are examples of the above proposition.

The cost $[C]$ of a quantity of meat varies as the price per pound $[P]$ if the weight $[W]$ is constant, and the cost varies as the weight if the price per pound is constant; hence, by the proposition, when both the weight and the price per pound change, the cost varies as the product of the weight and the price.

Thus, if $C \propto P$, when W is constant,

and $C \propto W$, when P is constant;

then $C \propto PW$, when both P and W change.

Again, the area $[A]$ of a triangle varies as the base $[B]$ when the height $[H]$ is constant; the area also varies as the height when the base is constant; hence, when both the base and the height change, the area will vary as the base and height jointly.

Thus, if $A \propto B$, when H is constant,

and if $A \propto H$, when B is constant;

then $A \propto BH$, when both B and H change.

Again, the pressure $[P]$ of a gas varies as the density $[D]$ when the absolute temperature $[T]$ is constant; the pressure also varies as the absolute temperature when the density is constant; hence, when both density and temperature change, the pressure will vary as the product of the density and temperature.

Thus $P \propto DT$.

213. Ex. 1. If $A \propto B$, and if also $A \propto C$; then will $B \propto C$.

For, since $A \propto B$, we have $A = mB$, where m is some constant.

And, since $A \propto C$, we have $A = nC$, where n is some constant.

Hence $B = \frac{n}{m} C$, where $\frac{n}{m}$ is some constant; and therefore $B \propto C$.

Ex. 2. If $C \propto WP$, then will $W \propto \frac{C}{P}$.

For, since $C \propto WP$, we have $C = m . WP$, where m is some constant.

Hence $W = \frac{1}{m} \frac{C}{P}$, where $\frac{1}{m}$ is some constant; therefore $W \propto \frac{C}{P}$.

Ex. 3. The pressure of a gas varies jointly as its density and its absolute temperature; also when the density is 1 and the temperature 300, the pressure is 15. What is the pressure when the density is 3 and the temperature is 320?

Since $P \propto TD$,

we have $P = mTD$,

where m is some constant.

Also, by the question,

$$15 = m \times 300 \times 1;$$

$$\therefore \quad m = \tfrac{1}{20}.$$

$$\therefore \quad P = \tfrac{1}{20} TD.$$

Hence, when D is 3 and T is 320, we have

$$P = \tfrac{1}{20} \times 320 \times 3$$
$$= 48.$$

EXAMPLES. XLVIII.

1. A varies as B, and A is 5 when B is 3: what is A when B is 5?

2. W varies inversely as P, and W is 4 when P is 15: what is W when P is 12?

3. If $x \propto y$ and $y \propto z$, then will $xz \propto y^2$.

4. If $x^2 \propto y$ and $z^2 \propto y$, then will $xz \propto y$.

5. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, then will $x \propto z$.

6. A varies as B and C jointly; also $A=4$ when $B=2$ and $C=6$: find the value of A when $B=2$ and $C=9$.

7. A varies as B and inversely as C; also $A=2$ when $B=3$ and $C=4$: find the value of B when $A=6$ and $C=3$.

8. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 10 feet is 314·159 square feet. What is the area of a circle whose radius is 12 feet?

9. The volume of a sphere varies as the cube of its radius, and the volume of a sphere whose radius is 1 foot is 4·188 cubic feet. What is the volume of a sphere whose radius is 3 feet?

10. The velocity of a falling body varies as the time during which it has fallen from rest, and the velocity at the end of two seconds is 64. What is the velocity at the end of five seconds?

11. The distance through which a heavy body falls from rest varies as the square of the time it falls, and a body falls through 144 feet in three seconds. How far does it fall in two seconds?

12. Given that the area of a circle varies as the square of its radius, shew that a circle of 5 feet radius is equal to the sum of a circle of 3 feet radius and another of 4 feet radius.

13. The volume of a gas varies as the absolute temperature and inversely as the pressure. Also when the pressure is 15 and the temperature 280 the volume is 1 cubic foot; what is the volume when the pressure is 20 and the temperature 300?

14. If a^2-b^2 varies as c^2, and if $c=2$ when $a=5$ and $b=3$; find the equation between a, b and c, and shew that b is a mean proportional between $a-2c$ and $a+2c$.

15. The volume of a right circular cone varies jointly as its height and the square of the radius of its base; and the volume of a cone 7 feet high with a base whose radius is 3 feet is 66 cubic feet. Find the volume of a cone 9 feet high with a base whose radius is 14 feet.

16. The volume of a sphere varies as the cube of its radius: if three spheres of radii 6, 8 and 10 inches respectively be melted and formed into a single sphere, find its radius.

17. The distance of the offing at sea varies as the square root of the height of the eye above the sea level, and the distance is 3 miles when the height is 6 feet: find the distance when the height is 50 yards.

MISCELLANEOUS EXAMPLES. V.

A.

1. If $\frac{2a}{3}=\frac{3b}{4}=\frac{c}{2}=1$, find the numerical value of

$$(a-b)^2+(b-c)^2+(c-a)^2.$$

2. Divide

$$9a^2b^3-12a^4b+3b^5+2a^3b^2+4a^5-11ab^4 \text{ by } 3b^3+4a^3-2ab^2.$$

3. Find the H.C.F. of $1-x-x^3+x^5$ and $1-x^4-x^6-x^7$.

4. Simplify $\frac{a^2-a^2b^{-2}-1+b^{-2}}{a+ab^{-1}+1+b^{-1}}$.

5. A person bought 15 ducks and 12 geese for £5. 5*s.*; and the prices were such that 2 more ducks could be bought for 18*s.*, than geese for £1. What was the price of a duck?

6. Shew that the difference of the roots of the equation $x^2-px+q=0$ is equal to the difference of the roots of the equation $x^2-3px+2p^2+q=0$.

7. What is the least integer which must be added to the terms of the ratio 3 : 4 to make it greater than the ratio 19 : 21?

8. Shew that, if $a+b$, $b+c$, $c+a$ are in continued proportion, then

$$b+c : c+a :: c-a : a-b.$$

B.

1. Find the value of

$$(xy+\sqrt{1-x^2}\,\sqrt{1-y^2})\div(x\sqrt{1-y^2}-y\sqrt{1-x^2}),$$

when $x=\frac{4}{5}$, $y=\frac{3}{5}$.

2. Multiply $x^2+(a-1)x+a+1$ by $(a-1)x-a^2-a-1$.

3. Find the factors of (i) $x^3-13x^2y+42xy^2$,

(ii) $(a+2b+3c)^2-4(a+b-c)^2$, and (iii) $x^2-4xy+4y^2-9$.

4. Simplify $\dfrac{x}{x-2y}+\dfrac{3x}{x+2y}+\dfrac{4xy}{4y^2-x^2}$.

5. Solve the equations:

(i) $\dfrac{10}{x}-\dfrac{3}{x+2}=\dfrac{10}{x+1}$.

(ii) $\left.\begin{aligned} x+y+\sqrt{(x+y)}&=12 \\ x^2-y^2&=21 \end{aligned}\right\}$.

6. If $x+\dfrac{1}{y}=1$ and $y+\dfrac{1}{z}=1$, prove that $z+\dfrac{1}{x}=1$, and that $xyz+1=0$.

7. Find the square root of

$$9x^6-12x^4y^2+30x^3y^3+4x^2y^4-20xy^5+25y^6.$$

8. Find two numbers such that their sum, their difference, and the sum of their squares are as 3 : 1 : 15.

9. If $a:b::c:d$, shew that

$$la+mb : lc+md :: \sqrt{a^3+b^2d} : \sqrt{ac^2+d^3}.$$

C.

1. Subtract $a-2(b-c)$ from $3\{b+2(a-c)-5(a-b)\}$.

2. Multiply $a+b+\dfrac{b^2}{a}+\dfrac{a^2}{b}$ by $a-b+\dfrac{b^2}{a}-\dfrac{a^2}{b}$.

3. Multiply $\dfrac{a^2+b^2}{9(a^2-b^2)}$ by $27(a-b)$, and divide

$$\frac{9a^2-16b^2}{a+b} \text{ by } \frac{3a-4b}{a^2-b^2}.$$

4. If $a=y+z$, $b=z+x$, $c=x+y$, then

$$a^2+b^2+c^2-bc-ca-ab=x^2+y^2+z^2-yz-zx-xy.$$

5. Find the square root of

$$a^{\frac{4}{3}}+4ay^{\frac{1}{5}}+10a^{\frac{2}{3}}y^{\frac{2}{5}}+12a^{\frac{1}{3}}y^{\frac{3}{5}}+9y^{\frac{4}{5}}.$$

6. If $x=2+\sqrt{2}$, prove that $(x-1)(x-2)=x$.

7. *A* and *B* start simultaneously from two towns to meet each other: *A* travels two miles an hour faster than *B* and they meet in seven hours: if *B* had travelled one mile an hour faster than he did and *A* at only half his previous pace, they would have met in nine hours. Find the distance between the towns.

8. If $a(y+z)=b(z+x)=c(x+y)$; then

$$\frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

D.

1. Shew that $\left(\dfrac{a^3-b^3}{a-b}\right)^2-\left(\dfrac{a^3+b^3}{a+b}\right)^2=4ab(a^2+b^2)$.

2. Divide

$$x^4-2bx^3-(a^2-b^2)x^2+2a^2bx-a^2b^2 \text{ by } x^2-(a+b)x+ab.$$

3. Find the L. C. M. of $x^2-7x+12$, $3x^2-6x-9$ and $2x^3-6x^2-8x$.

4. Simplify $\dfrac{\dfrac{y}{x}-\dfrac{y}{y+z}}{\dfrac{1}{xz}+\dfrac{1}{zy+z^2}}\left\{1+\dfrac{y^2+z^2-x^2}{2yz}\right\}$.

5. Solve the equations:

(i) $\dfrac{5x+7}{2}-\dfrac{2x-7}{3}=3x+14$.

(ii) $\left.\begin{aligned}3x^2-4xy&=20\\2x^2-5xy&=12\end{aligned}\right\}$.

6. The fore-wheel of a carriage makes 64 revolutions more than the hind-wheel in travelling one mile, and the sum of the revolutions made by each in a journey of 10 miles is 7040: find the circumference of each wheel.

7. Divide $2x^{\frac{3}{2}} - 17xy^{\frac{1}{2}} + 17x^{\frac{1}{2}}y - 15y^{\frac{3}{2}}$ by $2x^{\frac{1}{2}} - 15y^{\frac{1}{2}}$.

8. Prove that if the ratios $a : \alpha$, $b : \beta$ and $c : \gamma$ are all equal, then each is equal to the ratio $a + b + c : \alpha + \beta + \gamma$.

E.

1. Simplify the expression

$$2\left\{x - 2a - \tfrac{2}{3}(x - a) + a\right\} - \tfrac{1}{3}\left\{(x - a) - (a - b) + y - b\right\}.$$

2. Divide

$$2a^2x^2 - 2(3b - 4c)(b - c)y^2 + abxy \text{ by } ax + 2(b - c)y.$$

3. Find the factors of the following expressions:

(i) $4a^2b^4 - 16a^4b^2$, (ii) $x^2 - 26x - 87$, and (iii) $3x^2 - xy - 10y^2$.

4. Shew that

$$\frac{1}{(y - z)^2} + \frac{1}{(z - x)^2} + \frac{1}{(x - y)^2} = \left(\frac{1}{y - z} + \frac{1}{z - x} + \frac{1}{x - y}\right)^2.$$

5. A man bought a number of articles for £21. If he had got six more for the same money each would have cost 1*s*. 3*d*. less than it did; how many did he buy?

6. Multiply $\sqrt{(x - a)} + \sqrt{a} - \sqrt{x}$ by $\sqrt{(x - a)} - \sqrt{a} + \sqrt{x}$.

7. Shew that the mean proportional between $x^2 - \dfrac{1}{y^2}$ and $y^2 - \dfrac{1}{x^2}$ is $xy - \dfrac{1}{xy}$.

8. If $a : b :: c : d$, shew that

$$a^2 + b^2 : c^2 + d^2 :: (a + b)^2 : (c + d)^2,$$

and that

$$a^2 + b^2 + c^2 + d^2 : (a + b)^2 + (c + d)^2 :: (a + c)^2 + (b + d)^2 : (a + b + c + d)^2.$$

F.

1. Multiply $a - b + c - d$ by $a + b - c + d$.

2. Divide $a^3 + b^3 + c^3 + 3(b + c)(c + a)(a + b)$ by $a + b + c$.

3. Find the H.C.F. of $x^3 - 19x + 30$ and $5x^3 - 19x^2 + 36$. For what values of x will both expressions vanish?

4. Shew that $\left(\frac{a+b}{a-b}\right)^2 - \left(\frac{a-b}{a+b}\right)^2 = \frac{8ab(a^2+b^2)}{(a^2-b^2)^2}$.

5. Solve the equations:

(i) $\frac{6x-7}{9x+6} - \frac{5(x-1)}{12x+8} = \frac{1}{2}$.

(ii) $\left.\begin{aligned} 4x+6y &= 3 \\ 4x^2+9xy+9y^2 &= 11 \end{aligned}\right\}$.

6. Simplify $$\frac{2\sqrt{(\frac{1}{3})}+3\sqrt{(\frac{1}{2})}}{\frac{1}{2}\sqrt{(\frac{1}{3})}-\frac{1}{3}\sqrt{(\frac{1}{2})}},$$

and $$x^{\frac{1}{2}}y^{\frac{1}{3}} \times \left(\frac{x^{-\frac{2}{3}}}{y}\right) \div \{x^{-\frac{1}{6}}y^{-\frac{1}{4}}\}.$$

7. Find the square root of $$x^{12} - 6x^{10} + 13x^8 - 14x^6 + 10x^4 - 4x^2 + 1.$$

8. If $a : b :: c : d$; shew that $a^3c^2 : b^3d^2 :: a^5+c^5 : b^5+d^5$.

CHAPTER XXII.

ARITHMETICAL PROGRESSION.

214. A SERIES of quantities is said to be in *Arithmetical Progression* when the difference between any term and the preceding one is the same throughout the series.

Thus a, b, c, d, &c. are in Arithmetical Progression (A.P.) if $b-a=c-b=d-c$, &c.

The difference between each term of an A.P. and the preceding term is called the *common difference.*

The following are examples of arithmetical progressions:—

$$1,\quad 3,\quad 5,\quad 7,\ \&c.,$$
$$2,\quad 6,\quad 10,\quad 14,\ \&c.,$$
$$9,\quad 8,\quad 7,\quad 6,\ \&c.,$$

and
$$3,\quad -1,\quad -5,\quad -9,\ \&c.$$

In the first series the common difference is 2, in the second series it is 4, in the third it is -1, and in the last it is -4.

215. If the first term of an arithmetical progression be a, and the common difference d; then

the 2nd term will be $a+d$,
... 3rd .... $a+2d$,
... 4th .... $a+3d$,

and so on, the coefficient of d being always less by unity than the number giving the position of the term in the series.

Thus the nth term is

$$a + (n - 1)\,d.$$

We can therefore write down any term of an A. P. when the first term and the common difference are given. For example, in the A. P. whose first term is 5, and whose common difference is 3,

$$\text{the 9th term is } 5 + (9 - 1)\,3 = 29,$$

$$\text{and the 27th term is } 5 + (27 - 1)\,3 = 83.$$

216. When any *two* terms of an A. P. are given we can find the first term and the common difference, and therefore any other term of the series.

Suppose, for example, that the 10th term of an A. P. is 25, and the 15th term is 5.

Let the first term be a, and the common difference d.

Then the 10th term is $a + 9d$,

and the 15th term is $a + 14d$.

Hence $\qquad a + 9d = 25,$

and $\qquad a + 14d = 5.$

By subtraction $\qquad 5d = -20;$

$$\therefore \quad d = -4.$$

Then $\qquad a = 25 - 9d = 25 - 9\,(-4) = 61.$

Thus the series is 61, 57, 53, &c.

217. When three quantities are in arithmetical progression, the middle one is called the *Arithmetic Mean* of the other two.

Thus, if a, b, c are in A. P., b is the arithmetic mean of a and c.

By the definition of arithmetical progression, we have

$$b-a=c-b;$$

$$\therefore \quad b=\tfrac{1}{2}(a+c).$$

Thus *the arithmetic mean of any two quantities is half their sum.*

218. When any number of quantities are in arithmetical progression all the intermediate terms may be called *arithmetic means* of the two extreme terms.

Between any two given quantities any number of arithmetic means may be inserted.

For example, to insert four arithmetic means between 10 and 25.

We have to find an A. P. with *four* terms between 10 and 25, so that 10 is the first and 25 is the *sixth* term of an A. P.

Let d be the common difference.

Then the sixth term is

$$10+5d;$$

$$\therefore \quad 10+5d=25;$$

$$\therefore \quad d=3.$$

Thus the series is

$$10, \quad 13, \quad 16, \quad 19, \quad 22, \quad 25;$$

and the required arithmetic means between 10 and 25 are

$$13, \ 16, \ 19 \text{ and } 22.$$

We now consider the most general case, namely to insert n arithmetic means between a and b.

We have to find an A. P. with n terms between a and b, so that a is the first and b is the $(n+2)$th term of the A. P.

Let d be the common difference.

Then the $(n+2)$th term is $a+(n+1)d$;

$$\therefore \quad a+(n+1)d=b,$$

or

$$(n+1)d=b-a;$$

$$\therefore \qquad d=\frac{b-a}{n+1}.$$

Thus the series is

$$a, \quad a+\frac{b-a}{n+1}, \quad a+2\frac{b-a}{n+1} \ldots, \quad b;$$

and the arithmetic means required are

$$a+\frac{b-a}{n+1}, \quad a+2\frac{b-a}{n+1}, \quad a+3\frac{b-a}{n+1}, \ldots\ldots$$

EXAMPLES. XLIX.

1. Find the 30th terms of each of the following arithmetical progressions:

   (i) 3, 5, 7, &c.
   (ii) 1, 5, 9, &c.
   (iii) 12, 9, 6, &c.
   (iv) $\frac{1}{9}$, $\frac{1}{3}$, $\frac{5}{9}$, &c.
   (v) $a+b$, a, $a-b$, &c.

2. What is the 10th term of the A.P. whose first term is 7 and whose third term is 13?

3. What is the 12th term of the A.P. whose first term is 20 and whose sixth term is 10?

4. The 3rd term of an A.P. is 10, and the 14th term is 54: find the 20th term.

5. The 7th term of an A.P. is 5, and the 5th term is 7. What is the 12th term?

6. Which term of the series 5, 8, 11, &c. is 65?

7. Which term of the series $\frac{7}{6}$, $\frac{4}{3}$, $\frac{3}{2}$, &c. is 18?

8. Write down the arithmetic mean of (i) 7 and 13; (ii) 9 and -9, and (iii) $a+b$ and $a-b$.

9. Insert 5 arithmetic means between (i) 1 and $\frac{1}{7}$, (ii) 3 and 5, and (iii) a and $3a$.

10. Insert 10 arithmetic means between $5a-6b$ and $5b-6a$.

11. Insert 8 arithmetic means between $a-5b$ and $b-5a$.

12. If a, b, c, d are in A.P., shew that $a+d=b+c$.

13. The sum of the first and fourth terms of an A.P. is 19, and the sum of the third and sixth terms is 31. What is the first term?

14. If the same quantity be added to every term of an A.P., the sums will be in A.P.

15. If every term of an A.P. be multiplied by the same quantity the products will be in A.P.

16. If every alternate term of an A.P. be taken away, the remaining terms will be in A.P.

17. If between every two consecutive terms of an A.P. their arithmetic mean be inserted, the whole will form another arithmetical progression.

18. If four quantities are in A.P., the product of the first and fourth is always less than the product of the second and third.

219. We proceed to shew how to find the sum of any number of terms of an arithmetical progression.

Let a be the first term and d the common difference. Let n be the number of the terms whose sum is required, and let l be the last term.

Then, since l is the nth term, we have

$$l=a+(n-1)\,d.$$

Hence, if S be the required sum,

$$S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l.$$

Now write the series in the reverse order; then

$$S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a.$$

Hence, by addition of corresponding terms, we have

$$\begin{aligned}2S&=(a+l)+(a+l)+(a+l)+\ldots\text{ to }n\text{ terms}\\&=n\,(a+l);\end{aligned}$$

$$\therefore \quad S = \frac{n}{2}(a + l) \text{ ....(i).}$$

But $\quad a + l = a + a + (n-1)d = 2a + (n-1)d;$

$$\therefore \quad S = \frac{n}{2}\{2a + (n-1)d\} \text{ ....(ii).}$$

The formulae (i) and (ii) are both important and should be remembered.

It should be remarked that the formula (ii) gives the value of any one of the four quantities a, d, n and S when the other three are known.

Ex. 1. Find the sum of the first 20 terms of the series

$$5 + 8 + 11 + \&c.$$

Here $\quad a = 5, \quad d = 3, \quad n = 20;$

$$\begin{aligned}\therefore \quad S &= \frac{n}{2}\{2a + (n-1)d\} \\ &= \frac{20}{2}\{10 + 19 \times 3\} \\ &= 670.\end{aligned}$$

Ex. 2. Shew that the sum of any number of consecutive odd numbers, beginning with unity, is a square number.

The series of odd numbers is

$$1 + 3 + 5 + 7 + \dots .$$

Here $a = 1$ and $d = 2$; hence the sum of n terms is given by

$$\begin{aligned}S &= \frac{n}{2}\{2a + (n-1)d\} \\ &= \frac{n}{2}\{2 + (n-1)2\} \\ &= \frac{n}{2} \times 2n \\ &= n^2.\end{aligned}$$

Thus the sum of n consecutive odd numbers beginning with unity is n^2.

Ex. 3. The sum of 20 terms of an arithmetical progression is 410, and the first term is 30. What is the common difference?

We have

$$S = \frac{n}{2}\{2a + (n-1)d\},$$

where $S = 410, \quad a = 30$ and $n = 20$.

Hence

$$410 = \frac{20}{2}\{60 + 19d\}.$$

From which we find that $d = -1$.

Ex. 4. How many terms of the series $11 + 12 + 13 +$ &c. must be taken in order that the sum may be 410?

We have

$$S = \frac{n}{2}\{2a + (n-1)d\},$$

where $a = 11, \quad d = 1,$ and $S = 410$.

Hence

$$410 = \frac{n}{2}\{22 + (n-1)\};$$

$$\therefore \quad n^2 + 21n - 820 = 0,$$

that is

$$(n-20)(n+41) = 0;$$

$$\therefore \quad n = 20, \quad \text{or} \quad n = -41.$$

When a, s and d are given, n is to be found by solving a *quadratic* equation, and one of the roots of the equation is generally inapplicable, for any value of n which is not a *positive integer* is without meaning.

In the present case, -41 is to be rejected: thus the only value of n is 20.

Ex. 5. How many terms of the series $24 + 21 + 18 + \&c.$ must be taken in order that the sum may be 105?

We have

$$S = \frac{n}{2}\{2a + (n-1)d\},$$

where $a = 24, \quad d = -3, \text{ and } S = 105.$

Hence

$$105 = \frac{n}{2}\{48 + (n-1)(-3)\};$$

$$\therefore \quad 210 = n\{48 - 3n + 3\};$$

$$\therefore \quad n^2 - 17n + 70 = 0.$$

Hence $n = 7, \text{ or } n = 10.$

Since *both* values of n are positive integers, they are both answers to the question.

EXAMPLES. L.

Find the sum of the following series:

1. $2 + 4 + 6 + \dots$ to 20 terms.
2. $15 + 14\frac{1}{2} + 14 + \dots$ to 16 terms.
3. $1 + 2\frac{1}{4} + 3\frac{1}{2} \dots$ to 12 terms.
4. $-5 - 1 + 3 + \dots$ to 20 terms.
5. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \dots$ to 7 terms.
6. $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} - \dots$ to 61 terms.
7. $10 + \frac{29}{3} + \frac{28}{3} + \dots$ to 7 terms.
8. $\frac{7}{6} + 1 + \frac{5}{6} + \dots$ to 15 terms.
9. $3\frac{1}{3} + 2\frac{1}{2} + 1\frac{2}{3} + \dots$ to n terms.
10. $1\frac{1}{7} + 1\frac{17}{21} + 2\frac{10}{21} + \dots$ to n terms.
11. $\dfrac{1}{\sqrt{2}+1} + \sqrt{2} + \dfrac{1}{\sqrt{2}-1} + \dots$ to 7 terms.
12. $\dfrac{n-1}{n} + \dfrac{n-2}{n} + \dfrac{n-3}{n} + \dots$ to n terms.
13. $(a+b)^2 + (a^2+b^2) + (a-b)^2 + \dots$ to n terms.

14. The 4th term of an arithmetical progression is 15, and the 20th term is $23\frac{1}{2}$; find the sum of the first 20 terms.

15. Insert 29 arithmetic means between 5 and 50, and find their sum.

16. Find the sum of 20 terms of the series 3, 5, 7, &c. beginning at the seventh.

17. The sum of 10 terms of an A. P., whose first term is 2, is 155: what is the common difference?

18. The sum of 10 consecutive terms of the series 3, 8, 13, &c. is 705: which is the first of them?

19. Find the sum of 15 terms of an arithmetical progression of which the eighth is 6.

20. Shew that, if any odd number of quantities are in A. P., the first, the middle, and the last are also in A. P.

21. Shew that, if unity be added to the sum of any number of terms of the series 8, 16, 24, &c., the result will be the square of an odd number.

22. Find the sum of all the odd numbers between 100 and 200.

23. Find the sum of all the even numbers which are between 101 and 999.

24. Find the sum of all the numbers between 100 and 500 which are divisible by 3.

25. Find the sum of n terms of the series

$$n-\frac{1}{n}, \quad 3n-\frac{2}{n}, \quad 5n-\frac{3}{n}, \dots.$$

CHAPTER XXIII.

GEOMETRICAL PROGRESSION.

220. A SERIES of quantities is said to be in *Geometrical Progression* when the ratio of any term to the preceding one is the same throughout the series.

Thus a, b, c, d, &c. are in Geometrical Progression (G. P.) if $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$, &c.

The ratio of each term of a geometrical progression to the preceding term is called the *common ratio.*

The following are examples of geometrical progressions:

$$1, \quad 3, \quad 9, \quad 27, \text{ \&c.}$$
$$4, \quad 2, \quad 1, \quad \tfrac{1}{2}, \text{ \&c.}$$
and
$$\tfrac{2}{3}, \quad -1, \quad \tfrac{3}{2}, \quad -\tfrac{9}{4}, \text{ \&c.}$$

In the first series the common ratio is 3, in the second series it is $\frac{1}{2}$, and in the last it is $-\frac{3}{2}$.

221. If the first term of a G. P. be a, and the common ratio r; then

the 2nd term will be ar,
the 3rd .... ar^2,
the 4th .... ar^3,

and so on, the index of r being always less by unity than the number giving the position of the term in the series.

Thus the nth term is ar^{n-1}.

We can therefore write down any term of a G. P. when the first term and the common ratio are given. For example, in the G. P. whose first term is 5, and whose common ratio is 3,

$$\text{the 4th term is } 5 \times 3^3,$$

and $$\text{the 10th term is } 5 \times 3^9.$$

222. When any *two* terms of a G. P. are given, we can find the first term and the common ratio: the series will thus be completely determined.

Suppose, for example, that the 5th term of a G. P. is $\frac{8}{9}$, and the 7th term $\frac{32}{81}$.

Let the first term be a, and the common ratio r.

Then the 5th and the 7th terms are ar^4 and ar^6 respectively.

Hence $$ar^4 = \tfrac{8}{9},$$

and $$ar^6 = \tfrac{32}{81};$$

$\therefore$ by division, $$r^2 = \tfrac{4}{9};$$

$$\therefore \quad r = \pm \tfrac{2}{3}.$$

Then, since $$a \times \tfrac{16}{81} = \tfrac{8}{9},$$

$$a = \tfrac{9}{2}.$$

Thus the series is $\frac{9}{2}$, ± 3, 2, $\pm \frac{4}{3}$, &c.

223. When three quantities are in geometrical progression, the middle one is called the *geometric mean* of the other two.

Thus, if a, b, c are in G. P., b is the geometric mean of a and c. By the definition of geometric progression, we have

$$\frac{b}{a} = \frac{c}{b};$$

$$\therefore \quad b^2 = ac;$$

$$\therefore \quad b = \pm \sqrt{ac}.$$

Thus *the geometric mean of any two quantities is the square root of their product.*

It should be noticed that quantities which are in geometrical progression are in continued proportion [Art. 200], and that the geometric mean of two quantities is the same as their mean proportional.

224. When any number of quantities are in geometrical progression all the intermediate terms may be called *geometric means* of the two extreme terms.

Between two given quantities any number of geometric means may be inserted.

For example, to insert n geometric means between a and b.

We have to find a G. P. with n terms between a and b, so that a is the first and b is the $(n+2)$th term of the G. P.

Let r be the common ratio. Then the $(n+2)$th term is ar^{n+1};

$$\therefore \quad ar^{n+1} = b;$$

$$\therefore \quad r^{n+1} = \frac{b}{a};$$

$$\therefore \quad r = \sqrt[n+1]{\frac{b}{a}}.$$

Hence the required means are ar, ar^2, $ar^3 \ldots\ldots\ldots ar^n$, where $r = \sqrt[n+1]{\frac{b}{a}}$.

EXAMPLES. LI.

1. Find the 6th terms of each of the following geometrical progressions:

(i) $9, 3, 1$, &c.

(ii) $2, -3, \frac{9}{2}$, &c.

(iii) a^2, ab, b^2, &c.

2. What is the fifth term of the G. P. whose first term is 3 and whose third term is 4?

3. The 3rd term of a G. P. is 1, and the 6th term is $-\frac{1}{8}$: what is the 10th term?

4. Write down the geometric mean of (i) 4 and 9, (2) 7 and 252, and (iii) a^3b and ab^3.

5. Insert two geometric means between 1 and -8; also insert three geometric means between 12 and $\frac{3}{4}$, and four between $\frac{2}{3}$ and $\frac{81}{16}$.

6. If a, b, c, d are in G. P., shew that $ad=bc$.

7. If all the terms of a G. P. be multiplied by the same quantity, the products will be in G. P.

8. Shew that the reciprocals of the terms of a geometrical progression are in G. P.

9. If every alternate term of a G. P. be taken away, the remaining terms will be in G. P.

10. If between every two consecutive terms of a G. P. their geometric mean be inserted, the whole will form another geometrical progression.

11. Shew that the product of any two terms of a geometrical progression, which are respectively equally distant from the first and the last terms, is equal to the product of the first and last terms.

225. To find the sum of any number of terms of a geometrical progression.

Let a be the first term, and r the common ratio. Let n be the number of the terms whose sum is required, and let l be the last term.

Then, since l is the nth term, we have

$$l = ar^{n-1}.$$

Hence, if S be the required sum,

$$S = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}.$$

Multiply by r; then

$$Sr = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$S - Sr = a - ar^n;$$

that is $$S(1-r)=a(1-r^n);$$

$$\therefore\ S=\frac{a(1-r^n)}{1-r}.$$

The above formula may be written in another form: for, by substituting the value $l=ar^{n-1}$, we have

$$S=\frac{a-rl}{1-r}.$$

Ex. 1. Find the sum of 10 terms of the series 1, 2, 4, &c.

Here $$a=1,\ r=2,\ n=10.$$

$$\therefore\ S=a\frac{1-r^n}{1-r}$$

$$=\frac{1-2^{10}}{1-2}=2^{10}-1=1023.$$

Ex. 2. Find the sum of 6 terms of the series

$$2-3+\frac{9}{2}-\&c.$$

Here $a=2$, $r=\frac{-3}{2}$, and $n=6$.

$$\therefore\ S=a\frac{1-r^n}{1-r}$$

$$=2\frac{1-(-\frac{3}{2})^6}{1-(-\frac{3}{2})}=2\frac{1-\frac{3^6}{2^6}}{1+\frac{3}{2}}$$

$$=\tfrac{4}{5}\left(1-\frac{3^6}{2^6}\right)=-8\tfrac{5}{16}.$$

226. From the preceding article we have

$$S=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now if r is a *proper fraction*, whether positive or negative, the absolute value of r^n will decrease as n increases. [For example, if r is $\frac{1}{2}$, its successive powers will be $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c.] Also r^n can be made *as small as we please* by sufficiently increasing the value of n.

Hence, when r is numerically less than unity, the sum of the series can be made to differ from $\frac{a}{1-r}$ by as small a quantity as we please by taking n very great.

Thus the sum of an *infinite number* of terms of the geometrical progression $a + ar + ar^2 + \ldots$ in which r *is numerically less than unity*, is $\frac{a}{1-r}$.

Ex. 1. Find the sum of an infinite number of terms of the series

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

(ii) $9 - 6 + 4 - \ldots$

In (i) $$a = 1, \quad r = \tfrac{1}{2};$$

$$\therefore \; S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

In (ii) $$a = 9, \quad r = \frac{-6}{9} = -\tfrac{2}{3};$$

$$\therefore \; S = \frac{a}{1-r} = \frac{9}{1-(-\frac{2}{3})} = \frac{9}{\frac{5}{3}} = \frac{27}{5}.$$

Ex. 2. Find the value of $\cdot 23\dot{4}$.

$$\cdot 23\dot{4} = \cdot 23444 \ldots\ldots\ldots$$

$$= \frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \frac{4}{10^4} + \frac{4}{10^5} + \text{\&c.}$$

Now $\frac{4}{10^3}+\frac{4}{10^4}+\ldots$ to infinity $=\dfrac{\frac{4}{10^3}}{1-\frac{1}{10}}$

$$=\frac{10}{9}\times\frac{4}{10^3}=\frac{4}{900}.$$

Hence $\cdot 23\dot{4}=\frac{2}{10}+\frac{3}{100}+\frac{4}{900}=\frac{211}{900}.$

Ex. 3. Find the geometrical progression whose sum to infinity is 18, and whose second term is -8.

Let a be the first term, and r be the common ratio.

Then the second term is ar, and the sum to infinity is $\frac{a}{1-r}$.

Hence $$ar=-8,$$

and $$\frac{a}{1-r}=18.$$

Hence, by division, we have

$$r(1-r)=\frac{-8}{18};$$

$$\therefore\ r^2-r=\frac{4}{9};$$

$$\therefore\ r=-\frac{1}{3},\ \text{or}\ r=\frac{4}{3}.$$

If $$r=-\frac{1}{3},\quad a=\frac{-8}{r}=24.$$

Thus the series is $24,\ -8,\ \frac{8}{3}$, &c.

The value $r=\frac{4}{3}$ is inadmissible, for the sum of a geometrical progression is not given by the formula

$$S = \frac{a}{1-r},$$

except when r is numerically less than unity.

EXAMPLES. LII.

Find the sum of the following series:

1. $8+12+18+27+\ldots$ to n terms.
2. $12+9+6\frac{3}{4}+\ldots$ to 6 terms.
3. $a+\frac{a}{r}+\frac{a}{r^2}+\ldots$ to n terms.
4. $1\frac{1}{2}+1\frac{1}{3}+1\frac{5}{27}+\ldots$ to 8 terms.
5. $\frac{2}{9}+\frac{1}{3}+\frac{1}{2}+\ldots$ to 6 terms.
6. $12-9+6\frac{3}{4}-\ldots$ to infinity.
7. $4+1{\cdot}2+{\cdot}36+{\cdot}108+\ldots$ to infinity.
8. $4+{\cdot}8+{\cdot}16+\ldots$ to infinity.
9. $a^2+ab+b^2+\ldots$ to n terms.
10. $3-2+\frac{4}{3}-\ldots$ to infinity.

11. Find the sum of the successive powers of 2 from 2 up to 4096 inclusive.

12. Find, correct to four places of decimals, the sum to infinity of the series $1+\frac{1}{\sqrt{2}}+\frac{1}{2}+\ldots$

13. Find the geometric mean of $4x^2-12x+9$ and $9x^2+12x+4$.

14. Shew that the product of any odd number of terms of a G. P. will be equal to the n^{th} power of the middle term, n being the number of the terms.

15. Find the G. P. whose sum to infinity is 4, and whose second term is $\frac{3}{4}$.

16. Find the G. P. whose sum to infinity is 9, and whose second term is -4.

17. The sum of the first 10 terms of a certain G. P. is equal to 33 times the sum of the first 5 terms. What is the common ratio?

18. If the sum of a geometrical series to infinity is n times the first term, shew that the common ratio is $1-\frac{1}{n}$.

19. If the common ratio of successive terms of a G.P. be less than $\frac{1}{2}$, shew that each term is greater than the sum of all that follow it.

227. Sometimes we are not told the law which connects successive terms of a series; but when a certain number of the terms are given, the law can in simple cases be at once determined.

Suppose, for example, we have the series

$$3,\ 9,\ 15,\ \&c.$$

Here $9-3=6$, and $15-9=6$.

Thus the series is an arithmetical progression, whose common difference is 6.

Again, in the series

$$3,\ 9,\ 27,\ \&c.$$

Since $9-3=6$, and $27-9=18$, the series is *not* an arithmetical progression. We then see whether it satisfies the conditions for being a geometrical progression, namely $\frac{9}{3}=\frac{27}{9}$, which is the case. Thus the series is a geometrical progression, whose common ratio is 3.

EXAMPLES. LIII.

Sum the following series:

1. $3,\ 2{\cdot}7,\ 2{\cdot}4$, &c. to 21 terms.
2. $2,\ 18,\ 162$, &c. to 7 terms.
3. $1,\ {\cdot}2,\ {\cdot}04$, &c. to infinity.
4. $a,\ b,\ \frac{b^2}{a}$, &c. to n terms.
5. ${\cdot}3,\ {\cdot}03,\ {\cdot}003$, &c. to infinity.
6. $3+4{\cdot}3+5{\cdot}6+\ldots$ to 11 terms.

7. $a+\frac{4a+b}{3}+\frac{5a+2b}{3}+\ldots$ to 19 terms.

8. $\frac{1}{4}-\frac{1}{3}+\frac{4}{9}-$&c. to 8 terms.

9. $3\frac{1}{8}+1\frac{1}{4}+\frac{1}{2}+$&c. to infinity.

10. $2\frac{1}{2}+6\frac{1}{4}+15\frac{5}{8}+\ldots$ to 5 terms.

11. Sum the following series to 6 terms, and when possible to infinity:

(i) $2\frac{1}{2}+6\frac{1}{4}+10+\ldots$
(ii) $9+5+1+\ldots$
(iii) $4-3+\frac{9}{4}-\ldots$
(iv) $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\ldots$
(v) $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\ldots$

12. Shew that, if an odd number of quantities are in geometrical progression, the first, the middle and the last of them are also in geometrical progression.

13. The sum of the first 7 terms of an A.P. is 49, and the sum of the next 8 terms is 176: what is the series?

14. The arithmetic mean of the first and third terms of a geometrical progression is five times the second term: find the common ratio.

15. If the fourth term in an arithmetical progression is the geometric mean of the second and seventh terms; shew that the sixth term is the geometric mean of the second and fourteenth.

16. If P be the continued product of n terms of a geometrical progression whose first term is a and last term l, shew that

$$P^2=(al)^n.$$

17. The continued product of three numbers in G.P. is 216, and the sum of the products of them in pairs is 156: find the numbers.

18. Divide 25 into five parts which are in A.P., and which are such that the sum of the squares of the least and greatest of them is one less than the sum of the squares of the other three.

19. Insert between 6 and 16 two numbers such that the first three may be in A.P. and the last three in G.P.

20. If a, b, c be in geometrical progression, and x, y be the arithmetic means between a, b and b, c respectively, prove that

$$\frac{2}{b}=\frac{1}{x}+\frac{1}{y}, \quad \text{and} \quad 2=\frac{a}{x}+\frac{c}{y}.$$

CHAPTER XXIV.

HARMONICAL PROGRESSION.

228. A SERIES of quantities is said to be in *Harmonical Progression* when the difference between the first and the second of any three consecutive terms is to the difference between the second and the third as the first is to the third.

Thus a, b, c, d, &c. are in harmonical progression, if

$$a-b : b-c :: a : c,$$

$$b-c : c-d :: b : d,$$

and so on.

229. If a, b, c are in harmonical progression, we have by definition

$$a-b : b-c :: a : c.$$

Hence $$c(a-b)=a(b-c),$$

or $$ca-bc=ab-ac.$$

Divide by abc; then

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},$$

which shews that

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$$

are in arithmetical progression.

Thus, *if quantities are in harmonical progression, their reciprocals are in arithmetical progression.*

The above relation between quantities in harmonical progression is much more frequently employed than that given in the preceding article.

230. When three quantities are in harmonical progression, the middle one is called the *harmonic mean* of the other two.

If a, b, c are in harmonical progression,

$$\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}$$

are in arithmetical progression;

$$\therefore\quad \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b};$$

$$\therefore\quad \frac{2}{b}=\frac{1}{a}+\frac{1}{c};$$

$$\therefore\quad b=\frac{2ac}{a+c}.$$

Thus *the harmonic mean of any two quantities is twice their product divided by their sum.*

231. If we put A, G, H for the arithmetic, the geometric, and the harmonic means respectively of any two quantities a and b, we have

$$A=\tfrac{1}{2}(a+b),$$

$$G=\sqrt{ab},$$

and

$$H=\frac{2ab}{a+b}.$$

Hence

$$A\times H=\tfrac{1}{2}(a+b)\times\frac{2ab}{a+b}=ab;$$

$$\therefore\quad AH=G^2.$$

Hence G is the geometric mean of A and H.

Thus *the geometric mean of any two quantities is also the geometric mean of their arithmetic and harmonic means.*

232. Many questions concerning quantities in harmonical progression can be solved by considering the arithmetical progression whose terms are the reciprocals of the terms of the harmonical progression.

For example, to insert n harmonic means between any two quantities a and b.

Insert n arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$. These will be

$$\frac{1}{a}+\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\quad \frac{1}{a}+2\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\ \text{\&c.}$$

Thus

$$\frac{1}{a},\ \frac{1}{a}+\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\ \frac{1}{a}+2\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\ \dots,\ \frac{1}{b}$$

are in A. P.

That is

$$\frac{1}{a},\ \frac{(n+1)b+(a-b)}{(n+1)ab},\ \frac{(n+1)b+2(a-b)}{(n+1)ab},\ \dots\dots,\ \frac{1}{b}$$

are in A. P.

Hence the reciprocals of these are in H. P.

Thus the required harmonic means are

$$\frac{(n+1)ab}{(n+1)b+a-b},\ \frac{(n+1)ab}{(n+1)b+2(a-b)},\ \text{\&c.}$$

It should be remarked that no formula can be found giving the sum of any number of terms in harmonical progression.

233. The conditions that a, b, c may be in A. P., G. P., or H. P. respectively may be written

$$a-b : b-c :: a : a,$$

$$a-b : b-c :: a : b,$$

$$a-b : b-c :: a : c.$$

The first and third follow from the definitions of A. P. and H. P., and the second can be easily verified by multiplication.

Ex. 1. The second term of a harmonical progression is 2, and the fourth term is 6: find the series.

The second term of the corresponding arithmetical progression is $\frac{1}{2}$, and the fourth term is $\frac{1}{6}$. Hence, if a be the first term, and d the common difference, we have

$$a+d=\tfrac{1}{2}, \quad \text{and} \quad a+3d=\tfrac{1}{6},$$

whence $$a=\tfrac{2}{3}, \text{ and } d=-\tfrac{1}{6}.$$

Thus the arithmetical progression is

$$\tfrac{2}{3},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{6}, \text{ \&c.}$$

Hence the harmonical progression is

$$\tfrac{3}{2},\ 2,\ 3,\ 6, \text{ \&c.}$$

Ex. 2. Shew that $a+b$, $2b$ and $b+c$ will be in H. P. if a, b, c are in G. P.

The three quantities $a+b$, $2b$ and $b+c$ will be in H. P. if

$$\frac{1}{a+b}+\frac{1}{b+c}=\frac{2}{2b},$$

or if $$b(b+c)+b(a+b)=(a+b)(b+c);$$

that is, if $$b^2+bc+ab+b^2=ab+b^2+ac+bc,$$

or if $$b^2=ac;$$

and this is the case when a, b, c are in G. P.

EXAMPLES. LIV.

1. Shew that if the terms of a harmonical progression be all multiplied by the same quantity, the products will be in harmonical progression.

2. Insert 5 harmonic means between 1 and 7.

3. Insert 4 harmonic means between $\frac{2}{3}$ and $\frac{3}{2}$.

4. Shew that, if the arithmetic mean of two quantities is 1, their harmonic mean will be the square of their geometric mean.

5. Shew that, if a^2, b^2, c^2 are in arithmetic progression, $b+c$, $c+a$ and $a+b$ will be in harmonical progression.

6. Shew that if

$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b-a}+\frac{1}{b-c};$$

then $b=a+c$, or else a, b, c are in harmonical progression.

7. Shew that, if x, y, z be in H.P.,

$$(y+z-x)^2,\ (z+x-y)^2,\ (x+y-z)^2$$

will be in A.P.

8. Shew that, if x, y, z be in H.P., then will $\frac{x}{y+z}$, $\frac{y}{z+x}$, $\frac{z}{x+y}$ be also in H.P.

9. Shew that, if x, y, z be in H.P., then will

$$\frac{x}{y+z-x},\ \frac{y}{z+x-y},\ \frac{z}{x+y-z}$$

be also in H.P.

234. There are, besides the progressions, many other series the successive terms of which are formed according to simple laws. The following are examples of such series:

$$1^2+2^2+3^2+\ldots\ldots+n^2,$$

$$1\,.\,2+2\,.\,3+3\,.\,4+\ldots\ldots+n\,(n+1),$$

and

$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots\ldots+\frac{1}{n\,(n+1)}.$$

We shall conclude this chapter by shewing how to find the sum of n terms of some of these simple series.

Ex. 1. Find the sum of n terms of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$$

$$S_n=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{(n-1)n}+\frac{1}{n(n+1)}.$$

Now $\frac{1}{1.2}=\frac{1}{1}-\frac{1}{2}$; $\frac{1}{2.3}=\frac{1}{2}-\frac{1}{3}$; &c.

Hence

$$S_n=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right);$$

$$\therefore\ S_n=1-\frac{1}{n+1},$$

since all the terms except the first and the last destroy one another.

When n is infinite, $\frac{1}{n+1}$ is zero. Hence the sum of an infinite number of terms of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots \text{ is } 1.$$

Ex. 2. To find the sum of n terms of the series

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots$$

$$S_n=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\ldots\frac{1}{(n-1)n(n+1)}+\frac{1}{n(n+1)(n+2)}.$$

Since

$$\frac{1}{1.2.3}=\frac{1}{2}\left(\frac{1}{1.2}-\frac{1}{2.3}\right);\quad \frac{1}{2.3.4}=\frac{1}{2}\left(\frac{1}{2.3}-\frac{1}{3.4}\right),\ \&\text{c.},$$

we have

$$S_n=\frac{1}{2}\left(\frac{1}{1.2}-\frac{1}{2.3}\right)+\frac{1}{2}\left(\frac{1}{2.3}-\frac{1}{3.4}\right)+\dots$$
$$+\frac{1}{2}\left\{\frac{1}{(n-1)n}-\frac{1}{n(n+1)}\right\}.$$

Hence $$S_n=\frac{1}{2}\left\{\frac{1}{1.2}-\frac{1}{n(n+1)}\right\},$$

all the intermediate terms destroying one another.

When n is infinite, $\frac{1}{n(n+1)}$ is zero. Hence the sum of an infinite number of terms of the series

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\dots \text{ is } \frac{1}{4}.$$

Ex. 3. To find the sum of n terms of the series

$$1.2+2.3+3.4+\dots$$
$$S_n=1.2+2.3+3.4+\dots+n(n+1).$$

Now $1.2=\frac{1}{3}(1.2.3)$; $2.3=\frac{1}{3}(2.3.4-1.2.3)$; &c.

Hence

$$S_n=\tfrac{1}{3}(1.2.3)+\tfrac{1}{3}(2.3.4-1.2.3)+\tfrac{1}{3}(3.4.5-2.3.4)$$
$$+\tfrac{1}{3}\{(n-1)n(n+1)-(n-2)(n-1)n\}$$
$$+\tfrac{1}{3}\{n(n+1)(n+2)-(n-1)n(n+1)\}$$
$$=\tfrac{1}{3}n(n+1)(n+2),$$

since all the other terms cancel.

Thus $$S_n=\tfrac{1}{3}n(n+1)(n+2).$$

Ex. 4. To find the sum of n terms of the series

$$1^2+2^2+3^2+\dots$$
$$S_n=1^2+2^2+3^2+\dots+n^2.$$

Now $$n^2=n(n+1)-n;$$

$$\therefore S_n=1.2+2.3+3.4+\dots+n(n+1)$$
$$-1-2-3-\dots-n.$$

But $\quad 1 . 2 + 2 . 3 + \ldots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$, by Ex. 3.

Also $\quad 1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n+1)$.

Hence
$$S_n = \frac{1}{3} n(n+1)(n+2) - \frac{1}{2} n(n+1)$$
$$= \frac{1}{6} n(n+1)(2n+1).$$

Ex. 5. To find the sum of n terms of the series

$$1^3 + 2^3 + 3^3 + \ldots$$

$$S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3.$$

Now $\quad 4n^3 = \{n(n+1)\}^2 - \{(n-1)n\}^2$.

Change n into $n-1$; then

$$4(n-1)^3 = \{(n-1)n\}^2 - \{(n-2)(n-1)\}^2.$$

We have similarly

$$4(n-2)^3 = \{(n-2)(n-1)\}^2 - \{(n-3)(n-2)\}^2.$$

……

$$4 . 2^3 = (2 . 3)^2 - (1 . 2)^2,$$
$$4 . 1^3 = (1 . 2)^2.$$

Hence, by addition,

$$4\{1^3 + 2^3 + 3^3 + \ldots n^3\} = \{n(n+1)\}^2.$$

Hence $\quad 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4} n^2 (n+1)^2$.

This result may be expressed differently; for since

$$1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n+1),$$

we have

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2.$$

Thus *the sum of the cubes of the first n numbers is equal to the square of the sum of the numbers.*

Ex. 6. To find the sum of n terms of the series

$$1 + 2x + 3x^2 + 4x^3 + \ldots$$

Let $\quad S = 1 + 2x + 3x^2 + \ldots + (n+1)x^n$.

Then $\quad Sx = \quad x + 2x^2 + \ldots + nx^n + (n+1)x^{n+1}$.

Hence, by subtraction,

$$S(1-x)=1+x+x^2+\ldots+x^n-(n+1)x^{n+1}.$$

But $$1+x+x^2+\ldots+x^n=\frac{1-x^{n+1}}{1-x}.$$

Hence $$S(1-x)=\frac{1-x^{n+1}}{1-x}-(n+1)x^{n+1};$$

$$\therefore\ S=\frac{1-x^{n+1}}{(1-x)^2}-(n+1)\frac{x^{n+1}}{1-x}.$$

EXAMPLES. LV.

Find the sum of the following series to n terms, and when possible to infinity.

1. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\ldots$ 2. $\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+\ldots\ldots$

3. $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots\ldots$

4. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots\ldots$

5. $\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+3)(x+4)}+\ldots\ldots$

6. $\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+\ldots\ldots$

7. $2.4+4.6+6.8+\ldots\ldots$ 8. $1.2.3+2.3.4+3.4.5+\ldots\ldots$

9. $1.3+3.5+5.7+\ldots\ldots$ 10. $1^2+3^2+5^2+\ldots\ldots$

11. $2^2+4^2+6^2+\ldots\ldots$ 12. $a^2+(a+b)^2+(a+2b)^2+\ldots\ldots$

13. $a(a+b)+(a+b)(a+2b)+(a+2b)(a+3b)+\ldots\ldots$

14. Shew that the series of odd numbers 1, 3, 5, 7, &c. is such that the first is 1^3, the sum of the next two is 2^3, the sum of the next three is 3^3, and so on.

MISCELLANEOUS EXAMPLES. VI.

A.

1. Simplify
$(y+3)(y^2-1)-3(y+1)(y^2-9)+3(y-1)(y^2-9)-(y-3)(y^2-1)$.

2. Prove that

$$\frac{(2y-x)^3-(2x-y)^3}{3(y-x)}+\frac{(2y-x)^3+(2x-y)^3}{x+y}=10x^2-16xy+10y^2.$$

3. Find the H.C.F. of $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$. For what value of x will both expressions vanish?

4. Solve the equations:

(i) $3x+\frac{2}{y}-1=12x+\frac{5}{y}+14=\frac{1}{y}-2x-14.$

(ii) $x^2-(a+b+2c)x+(a+b+c)c=0.$

5. A man buys 9 oxen and 20 sheep for £230; by selling the oxen at a gain of 25 per cent., and the sheep at a loss of 20 per cent., he gains £35 altogether. Find the price he gave for each.

6. If α and β be the roots of the equation $x^2+4x+3=0$, shew that the equation whose roots are $\frac{\alpha+\beta}{\alpha}$ and $\frac{\alpha+\beta}{\beta}$ is $3x^2-16x+16=0$.

7. Multiply $a^{\frac{2}{3}}b^{-1}-2a^{\frac{1}{3}}b^{-\frac{1}{2}}+4-8a^{-\frac{1}{3}}b^{\frac{1}{2}}+16a^{-\frac{2}{3}}b$ by $a^{\frac{1}{3}}+2b^{\frac{1}{2}}$.

8. Rationalise the denominator of

$$\frac{5+4\sqrt{3}}{5-4\sqrt{3}},$$

and find

$$\sqrt{25+4\sqrt{34}}.$$

9. Sum the series $5-3\frac{3}{4}+2\frac{13}{16}-$ &c. to 8 terms, and find its limit when continued to infinity.

10. Insert 20 arithmetic means between 100 and 300, and find their sum.

B.

1. Simplify $3\{x-2(y-z)\}-[4y+2\{x-\overline{y-z}\}]$.

2. Find the factors of

$$mnx^2+m^2xy+n^2xy+mny^2,$$

and of

$$x^3-x^2y+xy^2-y^3.$$

3. If the sum of two numbers be equal to 4, shew that their difference is one-quarter of the difference of their squares.

4. Solve

(i) $\dfrac{a}{x+a}+\dfrac{b}{x+b}=1.$

(ii) $\sqrt{12x-3}+\sqrt{x+2}+\sqrt{7x-13}=0.$

5. Shew that $x^2-8x+22$ can never be less than 6.

6. Find the square root of $a^6+2a^3b^2-2a^3c+b^4-2b^2c+c^2$.

7. If $a:b::c:d$, shew that $a^2+b^2:c^2+d^2::(a+b)^2:(c+d)^2$.

8. Sum the following series:

(i) $5-1-7-$ &c. to 20 terms.

(ii) $2\frac{1}{2}+1+\frac{2}{5}+\ldots$ to infinity.

9. Find the sum of all the numbers which are less than 1000, and are divisible by 7.

10. A train travelling at the rate of $37\frac{1}{2}$ miles an hour passes a person walking on a road parallel to the railway in 6 seconds; it also meets another person walking at the same rate as the other but in the opposite direction, and passes him in 4 seconds. Find the length of the train.

C.

1. Find the value of $\dfrac{\sqrt{(y^2+z)}}{\sqrt{(z+y)}}-\dfrac{z-x(y-z)}{x-z(y-x)}$,

when $x=0, \quad y=1, \quad z=1\frac{1}{2}.$

2. Divide $4x^4-9x^2y^2+12xy^3-4y^4$ by $2x^2+3xy-2y^2$.

3. Shew that the square of the sum of two consecutive numbers is equal to four times their product increased by unity.

4. Find the L.C.M. of $x^2-6ax+9a^2$, $x^2-ax-6a^2$ and $3x^2-12a^2$.

5. Simplify $\dfrac{2}{a-2}+\dfrac{1}{1-a}+\dfrac{1}{a+1}-\dfrac{2}{a+2}.$

6. If 40 minutes would be saved in a journey by increasing the rate of the train by 5 miles an hour, and 1 hour would be lost by diminishing it by the same amount, find the rate of the train and the length of the journey.

7. Simplify $2+\sqrt{8}+\sqrt{2}-\sqrt{27}-\sqrt{12}+\sqrt{75}-\sqrt{(19+6\sqrt{2})}$.

8. Prove that any ratio is made nearer to unity by adding the same number to each of its terms. Also shew that $\dfrac{ma+nc}{mb+nd}$ is intermediate between $\dfrac{a}{b}$ and $\dfrac{c}{d}$.

9. Find the sum of the following series:
   (i) $21+15+9+\ldots$ to 8 terms.
   (ii) $5+\ 2+{\cdot}8+\ldots$ to infinity.

10. The arithmetic mean of two numbers is 17, and their geometric mean is 15: what are the numbers?

D.

1. Simplify $\{x(x+a)-a(x-a)\}\,\{x(x-a)-a(a-x)\}$.

2. Shew that $(n+1)^4-n^4=(2n+1)(2n^2+2n+1)$,
and that $$(a+b)^4-a^4-b^4=4ab(a+b)^2-2a^2b^2.$$

3. Simplify $$\left\{1-\frac{3x-20}{x^2-6x}\right\}\left\{1-\frac{8x-42}{x^2-5x}\right\}.$$

4. Solve the equations:
   (i) $\dfrac{3}{x}-\dfrac{5}{y}=21,\ \dfrac{5}{x}+\dfrac{7}{y}=-11.$
   (ii) $3x^2-4xy+2y^2=33,\ x^2-y^2=16.$

5. A number consists of two digits, of which that in the unit's place is the greater; the difference between the squares of the digits is equal to the number, and if the digits were inverted the number thus formed would be 7 times the sum of the digits. Find the number.

6. Find the equation whose roots are the cubes of the roots of the equation $x^2-4x+2=0$.

7. Find the square root of $x^6-2x^2+8+x^{-2}-8x^{-4}+16x^{-6}$.

8. If $a:b::c:d$, prove that $\dfrac{a^2+b^2}{c^2+d^2}=\left(\dfrac{a-b}{c-d}\right)^2$.

9. Sum the following series:
   (i) $\frac{1}{3}-\frac{1}{2}+\frac{3}{4}-\ldots$ to 10 terms.
   (ii) $9-6+4-\ldots$ to infinity.

10. Shew that if unity be added to the sum of any number of terms of the series 8, 16, 24, &c., the result will be the square of an odd number.

E.

1. Simplify $x^2 - 2xy + y^2 - (x^2 + xy + y^2) - \{x(-2y + x) + y^2\}$.

2. Divide $6a^4 + 4b^4 - a^3b + 13ab^3 + 2a^2b^2$ by $2a^2 + 4b^2 - 3ab$.

3. Find the highest common divisor and the lowest common multiple of $3a^3 - 3a^2b + ab^2 - b^3$ and $4a^2 - 5ab + b^2$.

4. Simplify
$$\frac{\dfrac{4}{x-1} + \dfrac{x-1}{x}}{\dfrac{1}{x-1} - \dfrac{1}{x}}.$$

5. If α and β are roots of $x^2 + mx + n = 0$, find $\alpha^2\beta + \beta^2\alpha$ in terms of m and n.

Test your result on the equation $x^2 - 3x + 2 = 0$.

6. Find the value of x for which $3x^2 + 5x + 3$ has its least possible value, and shew that the least value is $\frac{11}{12}$.

7. Shew that, if a, b, c, d be in continued proportion,
$$\left(\frac{a-b}{b-c}\right)^3 = \frac{a}{d}.$$

8. Find the square root of
$$4x^{\frac{4}{3}} - 12xy^{\frac{1}{2}} - 7x^{\frac{2}{3}}y + 24x^{\frac{1}{3}}y^{\frac{3}{2}} + 16y^2.$$

9. Sum the series:

(i) $-3 - 2 - 1 \ldots$ to n terms.

(ii) $1 - \frac{1}{2} + \frac{1}{4} \ldots$ to infinity.

10. A train 72 yards long passed another train 60 yards long, which was going in the opposite direction on a parallel line of rails, in 4 seconds. Had the first-mentioned train been travelling at twice its actual speed, the trains would have passed each other in 3 seconds. Find the number of miles per hour at which the trains were travelling.

F.

1. Simplify $(x + y + z)^2 - (-x + y + z)^2 + (x - y + z)^2 - (x + y - z)^2$.

2. Divide $a^3 - 3a^2b + 3ab^2 - b^3 - c^3$ by $a - b - c$.

3. Prove that, if
$$x = a + d, \qquad y = b + d, \qquad z = c + d,$$
then
$$x^2 + y^2 + z^2 - yz - zx - xy = a^2 + b^2 + c^2 - bc - ca - ab.$$

4. Simplify $\dfrac{1}{a-2x}+\dfrac{2a}{4x^2-a^2}-\dfrac{1}{a+2x}$.

5. Solve the equations:

(i) $\left.\begin{aligned} 2x+4y-3z&=22\\ 4x-2y+5z&=18\\ 5x+y-2z&=14 \end{aligned}\right\}.$

(ii) $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}$.

(iii) $\left.\begin{aligned} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}&=\dfrac{5}{2}\\ x^2+y^2&=90 \end{aligned}\right\}.$

6. Shew that, if any integer be put for x in the expression

$$x^6-4x^5+14x^4-32x^3+49x^2-60x+36,$$

the result will be a square number.

7. Reduce $\dfrac{1}{\sqrt{(12-\sqrt{140})}}$ to the best form for numerical calculations, and find its value to 4 places of decimals.

8. Shew that the ratio $a-x:a+x$ is greater or less than the ratio $a^2-x^2:a^2+x^2$, according as the ratio $x:a$ is greater or less than unity.

9. If $\dfrac{1}{x}+\dfrac{1}{y}$ varies inversely as $x+y$, then x^2+y^2 varies as xy.

10. If a, b and c are the sums of n, $2n$ and $3n$ terms respectively of a geometric progression, then $a^2+b^2=a(b+c)$.

CHAPTER XXV.

PERMUTATIONS AND COMBINATIONS.

235. THE different arrangements which can be made of a number of objects, the order in which the objects are arranged being taken into account, and any particular number being taken at a time, are called *permutations.*

Thus, suppose that we have three objects, represented by the three letters a, b, c.

If we take the three objects one at a time, we have the three permutations a, b, c.

If we take the objects two at a time, we have the six permutations ab, ba, ac, ca, bc and cb.

If we take the objects all together, we have also six permutations, namely abc, acb, bca, bac, cab and cba.

236. The number of permutations of n things taken r at a time is denoted by the symbol ${}_nP_r$.

237. *To find the number of permutations of* n *different things taken* r *at a time, where* r *is any integer not greater than* n.

Let the different objects be represented by the letters $a, b, c\ldots$, There are $n-1$ letters exclusive of a, and the number of permutations of these $n-1$ different letters taken $r-1$ at a time is ${}_{n-1}P_{r-1}$. If now we put a before each of these permutations we shall obtain all the permutations of the n letters taken r at a time in each of which a stands first, and the number of such permutations is ${}_{n-1}P_{r-1}$.

There are similarly ${}_{n-1}P_{r-1}$ permutations of the n letters r at a time in each of which b stands first; and so for c, d, &c.

Hence the whole number of permutations of n things taken r at a time is n times the number of permutations of $n-1$ things taken $r-1$ at a time. That is

$${}_{n}P_{r} = {}_{n-1}P_{r-1} \times n.$$

Since the above relation is true for all values of n and r, we may change n into $n-1$, and r into $r-1$; we then have

$${}_{n-1}P_{r-1} = {}_{n-2}P_{r-2} \times (n-1).$$

By repeating this change of n into $n-1$, and of r into $r-1$, we have in succession

$$\begin{aligned} {}_{n-2}P_{r-2} &= {}_{n-3}P_{r-3} \times (n-2), \\ {}_{n-3}P_{r-3} &= {}_{n-4}P_{r-4} \times (n-3), \\ \ldots\ldots &= \ldots\ldots \\ {}_{n-r+2}P_{2} &= {}_{n-r+1}P_{1} \times (n-r+2). \end{aligned}$$

Now the number of permutations of any number of things *one* at a time is obviously equal to the number of the things. Thus

$${}_{n-r+1}P_{1} = n-r+1.$$

By multiplying all the right-hand members and all the left-hand members of the above equalities, and cancelling all the common factors, we have

$${}_{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1) \ldots\ldots\ldots(\text{i}).$$

Thus *the number of permutations of* n *different things taken* r *at a time is the continued product of* r *successive integers of which* n *is the greatest.*

If the n things are to be taken all together, r is equal to n, and we have

$$ {}_nP_n = n(n-1)(n-2)\ldots 2\,.\,1 \ldots\ldots\ldots\ldots\ldots \text{(ii)}. $$

For example,

$$ {}_6P_4 = 6\,.\,5\,.\,4\,.\,3 = 360, $$

and $$ {}_6P_6 = 6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1 = 720. $$

The symbol $\underline{|n}$ is used to denote the continued product $n(n-1)(n-2)\ldots 3\,.\,2\,.\,1$. The symbol $\underline{|n}$ is read 'factorial n'. Sometimes $n\,!$ is used instead of $\underline{|n}$.

238. *To find the number of permutations of* n *things taken all together, when the things are not all different.*

Let there be n letters; and suppose p of them to be a's, q of them to be b's, r of them to be c's, and so on.

Let P be the required number of permutations.

If in any one of the actual permutations, we suppose that the a's are all changed into p letters different from each other and from all the rest; then, by changing only the arrangement of these p new letters, we should, instead of a single permutation, have $\underline{|p}$ different permutations.

Hence, if the a's were all changed into p letters different from each other and from all the rest, the b's, c's &c. being unaltered, there would be $P \times \underline{|p}$ permutations.

Similarly, if in any one of these new permutations we suppose the b's are all changed into q letters different from each other and from all the rest, we should obtain $\underline{|q}$ permutations by changing the arrangement of these q new letters. Hence the whole number of permutations would now be $P \times \underline{|p} \times \underline{|q}$.

By proceeding in this way we see that if all the letters were changed so that no two were alike, the total number of permutations would be

$$P \times \underline{|p} \times \underline{|q} \times \underline{|r} \ldots.$$

But the number of permutations of n different things is $\underline{|n}$.

Hence $$P \times \underline{|p} \times \underline{|q} \times \underline{|r} \times \ldots = \underline{|n};$$

$$\therefore \quad P = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r} \ldots}.$$

For example, the number of permutations of the six letters *aaabbc* taken all together is

$$\frac{\underline{|6}}{\underline{|3}\,\underline{|2}\,\underline{|1}} = 60.$$

Combinations.

239. The different selections which can be made from a number of objects, the order in which the objects are selected or arranged not being taken into account, and any particular number being taken at a time, are called *combinations.*

Thus the different combinations of the four letters a, b, c, d taken two at a time are ab, ac, ad, bc, bd, cd.

The number of combinations of n things r together is denoted by the symbol ${}_nC_r$.

240. *To find the number of combinations of* n *different things taken* r *together.*

Let the different objects be represented by the letters a, b, c ….

There are $n-1$ letters exclusive of a, and the number of combinations of these $n-1$ different letters taken $r-1$ at a time is ${}_{n-1}C_{r-1}$. If now we take a with each of these combinations, we shall obtain all the combinations of the n letters taken r at a time in each of which a occurs, and the number of such combinations is ${}_{n-1}C_{r-1}$.

There are similarly ${}_{n-1}C_{r-1}$ combinations of the n letters r at a time in each of which b occurs; and so for c, d, &c.

Now if we were to add together all the combinations which contain a, all those which contain b, and so on, we should count each combination r times, for each combination contains r letters; hence the whole number of combinations of the n things r together is ${}_{n-1}C_{r-1} \times n \div r$. Thus

$$ {}_{n}C_{r} = {}_{n-1}C_{r-1} \times \frac{n}{r}. $$

Since the above relation is true for all values of n and r, we may change n into $n-1$, and r into $r-1$; we then have

$$ {}_{n-1}C_{r-1} = {}_{n-2}C_{r-2} \times \frac{n-1}{r-1}. $$

By repeating this change of n into $n-1$ and of r into $r-1$, we have in succession

$$ {}_{n-2}C_{r-2} = {}_{n-3}C_{r-3} \times \frac{n-2}{r-2}, $$

$$ {}_{n-3}C_{r-3} = {}_{n-4}C_{r-4} \times \frac{n-3}{r-3}, $$

$$ \dots\dots\dots = \dots\dots\dots $$

$$ {}_{n-r+2}C_{2} = {}_{n-r+1}C_{1} \times \frac{n-r+2}{2}. $$

Also the number of combinations of any number of things *one* at a time is obviously equal to the number of the things. Thus

$$_{n-r+1}C_1 = \frac{n-r+1}{1}.$$

Multiply all the right-hand members and all the left-hand members of the above equalities, and cancel the common factors: we then have

$$_nC_r = \frac{n}{r} \times \frac{n-1}{r-1} \times \frac{n-2}{r-2} \times \dots \frac{n-r+1}{1}$$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}}.$$

Thus *the number of combinations of* n *different things* r *together is the quotient obtained by dividing the continued product of* r *successive integers the greatest of which is* n *by the continued product of* r *successive integers the greatest of which is* r.

For example,

$$_8C_3 = \frac{8 \,.\, 7 \,.\, 6}{1 \,.\, 2 \,.\, 3} = 56;$$

also $$_8C_5 = \frac{8 \,.\, 7 \,.\, 6 \,.\, 5 \,.\, 4}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} = 56.$$

241. In Articles 237 and 240, we found $_nP_r$ and $_nC_r$ independently of one another. But, since every combination of r different things would give rise to $\underline{|r}$ permutations if the order of its letters were altered in every possible way, we have

$$_nC_r \times \underline{|r} = {}_nP_r.$$

This gives the number of combinations of n different things r together in terms of the number of permutations.

242. We found in Art. 240, that

$$ {}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\,\ldots\,r}; $$

$$ \therefore\ {}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,\ldots\,r} \times \frac{\underline{|n-r}}{\underline{|n-r}} $$

$$ = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}, $$

since $$ n(n-1)\ldots(n-r+1) \times \underline{|n-r} = \underline{|n}. $$

This formula is frequently employed.

243. From the formula in the last Article, we have

$$ {}_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}; $$

$$ \therefore\ {}_nC_r = {}_nC_{n-r}, $$

so that *the number of combinations of* n *things* r *together is equal to the number of combinations of n things* n − r *together.*

The above proposition follows however at once from the fact that whenever we take r out of n things we must leave $n-r$; and therefore the number of different ways of taking r things must be just the same as the number of different ways of leaving or taking $n-r$ things.

244. The total number of combinations of $(n+1)$ things r together can be divided into two groups according as they do or do not contain a certain particular letter. The number which do *not* contain the particular letter is the number of ways in which r of the remaining n things can be taken, which is ${}_nC_r$, and the number which do contain the letter is the number of ways in which $(r-1)$ of the remaining n things can be taken, which is ${}_nC_{r-1}$.

Thus $${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}.$$

This is an important proposition, which we may also prove from the general formulae as follows.

$$\begin{aligned} {}_nC_r + {}_nC_{r-1} &= \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r} \\ &\qquad + \frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots(r-1)} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+2)}{\underline{|r}}[n-r+1+r] \\ &= \frac{(n+1)n(n-1)(n-2)\ldots(n-r+2)}{\underline{|r}} \\ &= {}_{n+1}C_r. \end{aligned}$$

245. From Art. 240 we have

$${}_nC_r = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \ldots \times \frac{n-r+1}{r};$$

$$\therefore \quad {}_nC_r = {}_nC_{r-1} \times \frac{n-r+1}{r}.$$

Hence ${}_nC_r \gtreqless {}_nC_{r-1}$, according as $n-r+1 \gtreqless r$, that is, according as $r \lesseqgtr \frac{1}{2}(n+1)$.

Thus the number of combinations of n things increases with r so long as r is less than $\frac{1}{2}(n+1)$.

If then n be even, ${}_nC_r$ is greatest when $r = \dfrac{n}{2}$.

If n be odd, ${}_nC_r \gtrless {}_nC_{r-1}$ when $r \lessgtr \frac{1}{2}(n+1)$, and

$${}_nC_r = {}_nC_{r-1} \text{ when } r = \tfrac{1}{2}(n+1).$$

Thus, when n is odd, ${}_nC_r$ is greatest when r is $\frac{1}{2}(n-1)$, or $\frac{1}{2}(n+1)$.

For example, when $n = 8$,

$$ {}_nC_r = \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{5} \times \dots, $$

and ${}_nC_r$ is greatest when $r = 4$.

Again, when $n = 9$,

$$ {}_nC_r = \frac{9}{1} \times \frac{8}{2} \times \frac{7}{3} \times \frac{6}{4} \times \frac{5}{5} \times \frac{4}{6} \times \dots, $$

and ${}_nC_r$ is greatest when r is 4 or 5.

EXAMPLES. LVI.

1. Find ${}_{15}P_3$, ${}_{10}P_4$ and ${}_8P_8$.

2. Shew that the number of permutations of 10 things taken 4 together is equal to the number of permutations of 7 things taken all together.

3. Find the number of permutations of all the letters of each of the words *success*, *mississippi* and *algebraically*.

4. In how many ways can the nine letters *aaaabbbcc* be arranged in a row?

5. How many different numbers can be formed with the figures of 34567; and how many with the figures of 1112233?

6. Find ${}_nP_n$, having given that ${}_nP_5 = {}_nP_4 \times 2$.

7. Find n, having given that ${}_nP_6 = 12 \times {}_nP_4$.

8. Find ${}_{16}C_4$, ${}_{15}C_5$, and ${}_6C_5$.

9. If ${}_nC_5 = {}_nC_{10}$, find ${}_nC_2$.

10. If ${}_{2n}C_3 = 12 \times {}_nC_2$, find n.

11. Shew that in ${}_8C_4$ the number of combinations in which a particular thing occurs is equal to the number in which it does not occur.

12. Shew that in ${}_{2n}C_n$ the number of combinations in which a particular thing occurs is equal to the number in which it does not occur.

13. Shew that in ${}_{12}C_4$ the number of the combinations in which a particular thing occurs is one-third of the whole number of the combinations.

14. Shew that in ${}_{3n}C_n$ the number of the combinations in which a particular thing occurs is one-third of the whole number of the combinations.

15. There are 10 candidates for 6 vacancies in a committee: in how many ways can a person vote for 6 of the candidates?

16. In how many ways can a cricket eleven be chosen out of fourteen players?

17. Out of 5 black balls and 4 white ones, in how many ways can 4 balls be chosen, two of the four being black and the other two being white?

18. In how many ways could 2 ladies and 2 gentlemen be chosen to make a set at tennis from a party of 4 ladies and 6 gentlemen?

19. In how many ways could 2 ladies and 2 gentlemen be chosen to make a set at tennis from a party of 6 ladies and 8 gentlemen?

20. There are four letters and four directed envelopes: in how many ways can all the letters be put into the wrong envelopes?

CHAPTER XXVI.

THE BINOMIAL THEOREM.

246. WE have already proved that the product of any two compound expressions is the sum of all the partial products obtained by multiplying any term of one expression by any term of the other.

To find the continued product of three expressions we must multiply each of the terms in the product of the first two expressions by each of the terms in the third; hence the continued product is the sum of all the partial products which can be obtained by multiplying together any term of the first, any term of the second and any term of the third.

And similarly, the continued product of any number of expressions is the sum of all the partial products which can be obtained by multiplying together any term of the first, any term of the second, any term of the third, &c.

For example, if we take a letter from each of the factors of

$$(a+b)(a+b)(a+b),$$

and multiply the three together, we shall obtain a term of the continued product; and if we do this in every possible way we shall obtain all the terms of the continued product.

We can take a every time, and we can do this in only one way; hence a^3 is a term of the continued product.

We can take a twice and b once, and we can do this in *three* ways, for the b can be taken from either of the three binomial factors; hence we have $3a^2b$.

We can take a once and b twice, and we can do this also in *three* ways; hence we have $3ab^2$.

Finally, we can take b every time, and this can be done in only one way; hence we have b^3.

Thus the continued product is

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

so that $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

247. Suppose we have n factors each of which is $a+b$.

If we take a letter from each of the factors of

$$(a+b)(a+b)(a+b)\ldots\ldots$$

and multiply them all together, we shall obtain a term of the continued product; and if we do this in every possible way we shall obtain all the terms of the continued product.

Now we can take the letter a every time, and this can be done in only one way: hence a^n is a term of the product.

The letter b can be taken once, and a the remaining $(n-1)$ times, and the number of ways in which one b can be taken is the number of ways of taking 1 out of n things, so that the number is ${}_nC_1$: hence we have

$${}_nC_1 \cdot a^{n-1}b.$$

Again, the letter b can be taken twice, and a the remaining $(n-2)$ times, and the number of ways in which two b's can be taken is the number of ways of taking 2 out of n things, so that the number is ${}_nC_2$: hence we have

$${}_nC_2 \cdot a^{n-2}b^2.$$

And, in general, b can be taken r times (where r is any positive integer not greater than n) and a the remaining $(n-r)$ times, and the number of ways in which r b's can be taken is the number of ways of taking r out of n things, so that the number is ${}_nC_r$: hence we have

$${}_nC_r \,.\, a^{n-r} b^r.$$

The letter b can be taken every time in only one way: hence we have the term b^n, which agrees with the result obtained by putting $r=n$ in ${}_nC_r \,.\, a^{n-r} b^r$, since ${}_nC_n = 1$.

Thus $\quad (a+b)(a+b)(a+b) \ldots$ to n factors

$$= a^n + {}_nC_1 \,.\, a^{n-1} b + {}_nC_2\, a^{n-2} b^2 + \ldots$$
$$+ {}_nC_r\, a^{n-r} b^r + \ldots + b^n.$$

Hence, when n is any positive integer, we have

$$(a+b)^n = a^n + {}_nC_1 \,.\, a^{n-1} b + {}_nC_2 \,.\, a^{n-2} b^2 + \ldots$$
$$+ {}_nC_r\, a^{n-r} b^r + \ldots + b^n.$$

The above formula is called the *Binomial Theorem.*

The series on the right is called the *expansion* of $(a+b)^n$.

If we substitute the known values [Art. 240] of ${}_nC_1$, ${}_nC_2$, &c. in the series on the right we obtain the form in which the theorem is usually written, namely

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}\, a^{n-2}b^2 + \ldots$$
$$+ \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}\, a^{n-r} b^r + \ldots + b^n.$$

248. Any term of the expansion of $(a+b)^n$ will be found by giving a suitable value to r in $\dfrac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}\, a^{n-r} b^r$.

On this account, the above is called the *general term* of the expansion.

249. If we put $n=2$ in the formula for the expansion of a binomial, we have

$$(a+b)^2 = a^2 + 2ab + b^2.$$

If we put $n=3$, we have

$$\begin{aligned}(a+b)^3 &= a^3 + 3a^2b + \frac{3 \cdot 2}{1 \cdot 2}\, ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

If we put $n=4$, we have

$$\begin{aligned}(a+b)^4 &= a^4 + 4a^3b + \frac{4 \cdot 3}{1 \cdot 2}\, a^2b^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}\, ab^3 + b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.\end{aligned}$$

If we put $2x$ for a, and $-3y$ for b, and $n=5$, we have

$$\begin{aligned}(2x-3y)^5 &= (2x)^5 + 5\,(2x)^4(-3y) + \frac{5 \cdot 4}{1 \cdot 2}\,(2x)^3(-3y)^2 \\ &+ \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}\,(2x)^2(-3y)^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}\,(2x)(-3y)^4 + (-3y)^5 \\ &= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5.\end{aligned}$$

250. The Binomial Theorem may also be proved in the following manner.

We have to prove that, when n is any positive integer,

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}\, a^{n-2}b^2 + \ldots \\ &\qquad + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\, a^{n-r}b^r + \ldots + b^n,\end{aligned}$$

or that

$$\begin{aligned}(a+b)^n &= a^n + {}_nC_1\, a^{n-1}b + {}_nC_2\, a^{n-2}b^2 + \ldots \\ &\qquad + {}_nC_{r-1}\, a^{n-r+1}b^{r-1} + {}_nC_r\, a^{n-r}b^r + \ldots + b^n.\end{aligned}$$

Now if we *assume* that the theorem is true when the index is n and multiply by another factor $a+b$, we have, when the terms in the product which contain the same powers of a and b are collected,

$$(a+b)^{n+1} = a^{n+1} + (1 + {}_nC_1)\, a^n b + ({}_nC_1 + {}_nC_2)\, a^{n-1} b^2 + \ldots$$
$$+ ({}_nC_{r-1} + {}_nC_r)\, a^{n-r+1} b^r + \ldots + b^{n+1}.$$

Now $\qquad 1 + {}_nC_1 = 1 + n = {}_{n+1}C_1,$

$$ {}_nC_1 + {}_nC_2 = n + \frac{n(n-1)}{2} = \frac{(n+1)n}{2} = {}_{n+1}C_2,$$

and, for any value of r,

$$ {}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r\text{*}. \qquad \text{[Art. 244.]}$$

Hence $(a+b)^{n+1} = a^{n+1} + {}_{n+1}C_1 \cdot a^n b + {}_{n+1}C_2 \cdot a^{n-1} b^2 + \ldots$
$$+ {}_{n+1}C_r \cdot a^{n+1-r} b^r + \ldots + b^{n+1}.$$

Thus, *if* the theorem be true for any value of n, it will be true for the next greater value.

Now the theorem is obviously true when $n = 1$. Hence it must be true when $n = 2$; and being true when $n = 2$, it must be true when $n = 3$; and so on indefinitely. The theorem is therefore true for *all* positive integral values of n.

The above method of proof is frequently employed in Algebra, and is called the method of *Induction*.

251. In the expansion of $(a+b)^n$ by the binomial theorem, the $r+1$th term from the beginning and the $r+1$th term from the end are respectively

$$ {}_nC_r a^{n-r} b^r \text{ and } {}_nC_{n-r} a^r b^{n-r}.$$

But ${}_nC_r = {}_nC_{n-r}$, for all values of r. [Art. 243.]

Hence in the expansion of $(a+b)^n$, *the coefficients of any two terms equidistant from the beginning and the end are the same.*

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* In writing out a proof of the Binomial Theorem it would be well to insert Art. 244 at this stage. This is the fundamental point in the proof of the Binomial theorem, but it is frequently omitted altogether.

252. If in the formula of Art. 247 we put $a=1$ and $b=x$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \ldots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}x^r + \ldots + x^n.$$

This is the most convenient form of the Binomial Theorem, and the one which is generally employed.

It should be noticed that the above form of the Binomial Theorem includes all cases; if, for example, we want to find $(a+b)^n$, we have

$$(a+b)^n = \left\{a\left(1+\frac{b}{a}\right)\right\}^n = a^n\left(1+\frac{b}{a}\right)^n = a^n\left(1+n\frac{b}{a}+\&c.\right)$$
$$= a^n + na^{n-1}b + \&c.$$

EXAMPLES. LVII.

Write out the following expansions:

1. $(x+a)^6$.	2. $(1-x^2)^5$.	3. $(3x-2y)^4$.
4. $(2a+3a^2)^4$.	5. $(2x^2-1)^6$.	6. $(y-x)^7$.

7. Find the third term of $(a-3b)^{10}$.

8. Find the fifth term of $(2x-x^2)^{12}$.

9. Find the sixth term of $(2a-\frac{1}{2})^8$.

10. Find the seventh term of $(1-x)^{10}$.

11. Find the eighteenth term of $(1+x)^{20}$.

12. Find the twenty-first term of $(1-x)^{22}$.

13. Expand $(a+\sqrt{b})^4+(a-\sqrt{b})^4$.

14. Expand $(a+\sqrt{b})^6+(a-\sqrt{b})^6$.

15. Expand $(a+\sqrt{b})^8+(a-\sqrt{b})^8$.

16. Find the middle term of $(1+x)^8$.

17. Find the middle term of $(1+x)^{10}$.

18. Find the middle term of $(2x-3y)^8$.

19. Find the term in the expansion of $(a+x)^8$ which has the greatest coefficient. [Art. 245.]

20. Find the two terms in the expansion of $(a+x)^9$ which have the greatest coefficients.

21. Write down the first three and the last three terms of the expansion of $(a+x)^{12}$.

22. Shew that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

23. Shew that the middle term of $(1+x)^{2n}$ is $\dfrac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{\underline{|n}}\, 2^n x^n$.

24. Employ the binomial theorem to find 99^4 and 999^3.

253. We now proceed to consider some properties of the coefficients of a binomial expansion.

It is often convenient to write the binomial theorem in the following form

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_r x^r + \ldots + c_n x^n,$$

where

$$c_0 = c_n = 1, \quad c_1 = c_{n-1} = n, \quad \text{and} \quad c_r = c_{n-r} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

254. Put $x = 1$ in the formula of Art. 253; and we have

$$2^n = c_0 + c_1 + c_2 + \ldots + c_r + \ldots c_n.$$

Thus *the sum of the coefficients in the expansion of* $(1+x)^n$ *is* 2^n.

Again, put $x = -1$; and we have

$$(1-1)^n = c_0 - c_1 + c_2 - c_3 + \ldots;$$

$$\therefore\ 0 = (c_0 + c_2 + c_4 + \ldots) - (c_1 + c_3 + c_5 + \ldots).$$

Thus *the sum of the coefficients of the odd terms of a binomial expansion is equal to the sum of the coefficients of the even terms.*

Ex. 1. Shew that

$$c_1 + 2c_2 + 3c_3 + \dots + rc_r + \dots + nc_n = n2^{n-1}.$$

$$\begin{aligned} c_1 + 2c_2 + 3c_3 + \dots + rc_r + \dots + nc_n \\ &= n + 2 \,.\, \frac{n(n-1)}{1\,.\,2} + 3 \,.\, \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} + \dots \\ &\qquad + r\frac{|\underline{n}}{|\underline{r}\,|\underline{n-r}} + \dots + n \\ &= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1\,.\,2} + \dots \right. \\ &\qquad \left. + \frac{|\underline{n}}{|\underline{r-1}\,|\underline{n-r}} + \dots + 1\right\} \\ &= n(1+1)^{n-1} \\ &= n2^{n-1}. \end{aligned}$$

Ex. 2. Shew that

$$c_0 - \tfrac{1}{2}c_1 + \tfrac{1}{3}c_2 - \dots + (-1)^n \frac{c_n}{n+1} = \frac{1}{n+1}.$$

$$c_0 - \tfrac{1}{2}c_1 + \tfrac{1}{3}c_2 - \tfrac{1}{4}c_3 + \dots + (-1)^n \frac{c_n}{n+1}$$

$$= 1 - \tfrac{1}{2} \,.\, n + \tfrac{1}{3}\frac{n(n-1)}{1\,.\,2} - \tfrac{1}{4}\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3} + \dots + (-1)^n \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left\{(n+1) - \frac{(n+1)n}{1\,.\,2} + \frac{(n+1)n(n-1)}{1\,.\,2\,.\,3} - \dots + (-1)^n\right\}$$

$$= \frac{1}{n+1} - \frac{1}{n+1}\left\{1 - (n+1) + \frac{(n+1)n}{1\,.\,2} - \dots + (-1)^{n+1}\right\}$$

$$= \frac{1}{n+1} - \frac{1}{n+1}(1-1)^{n+1}$$

$$= \frac{1}{n+1}.$$

255. Since $c_r = c_{n-r}$, we may write the binomial theorem in either of the following ways

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_rx^r + \ldots c_nx^n,$$

$$\text{or}\quad (1+x)^n = c_n + c_{n-1}x + c_{n-2}x^2 + \ldots c_{n-r}x^r + \ldots + c_0x^n.$$

The coefficient of x^n in the product of the two series on the right is equal to

$$c_0^2 + c_1^2 + c_2^2 + \ldots c_n^2.$$

Hence* $c_0^2 + c_1^2 + \ldots c_r^2 + \ldots c_n^2$ is equal to the coefficient of x^n in $(1+x)^n \times (1+x)^n$, that is in $(1+x)^{2n}$, and this coefficient is $\dfrac{\underline{|2n}}{\underline{|n}\,\underline{|n}}$.

Hence *the sum of the squares* of the coefficients in the expansion of $(1+x)^n$ is equal to $\dfrac{\underline{|2n}}{\underline{|n}\,\underline{|n}}$.

256. By means of the binomial theorem we can find the expansion of $(a+b+c)^n$. Thus

$$(a+b+c)^n = \{a + (b+c)\}^n$$

$$= a^n + \ldots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\,a^r(b+c)^{n-r} + \ldots$$

$$\text{Also}\quad (b+c)^{n-r} = \ldots + \frac{\underline{|n-r}}{\underline{|s}\,\underline{|n-r-s}}\,b^sc^{n-r-s} + \ldots$$

Hence the coefficient of $a^rb^sc^t$, where r, s, t are any integers such that $r+s+t=n$, is

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \times \frac{\underline{|n-r}}{\underline{|s}\,\underline{|t}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}}.$$

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* See Art. 266.

Thus the *general term* of the expansion of $(a+b+c)^n$ is

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}}\,a^r b^s c^t.$$

For example, the coefficient of abc in the expansion of $(a+b+c)^3$ is

$$\frac{\underline{|3}}{\underline{|1}\,\underline{|1}\,\underline{|1}} = 6.$$

Also the coefficient of a^4b^2c in the expansion of $(a+b+c)^7$ is

$$\frac{\underline{|7}}{\underline{|4}\,\underline{|2}\,\underline{|1}} = 105.$$

EXAMPLES. LVIII.

In the following examples $c_0, c_1, c_2 \ldots$ are the coefficients of $x^0, x^1, x^2 \ldots$ in the expansion of $(1+x)^n$.

1. Prove that $c_1 - 2c_2 + 3c_3 - \ldots + (-1)^{n-1} nc_n = 0$.

2. Prove that $c_0 - 2c_1 + 3c_2 - \ldots + (-1)^n (n+1)c_n = 0$.

3. Prove that $c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \ldots + \dfrac{1}{n+1}\,c_n = \dfrac{2^{n+1}-1}{n+1}$.

4. Prove that $\dfrac{c_1}{c_0} + 2\dfrac{c_2}{c_1} + 3\dfrac{c_3}{c_2} + \ldots + n\,\dfrac{c_n}{c_{n-1}} = \frac{1}{2}n(n+1)$.

5. Prove that

$$c_0 + c_1x + 2c_2x^2 + \ldots + rc_rx^r \ldots + nc_nx^n = 1 + nx(1+x)^{n-1}.$$

6. Prove that $c_0c_1 + c_1c_2 + c_2c_3 + \ldots + c_{n-1}c_n = \dfrac{\underline{|2n}}{\underline{|n+1}\,\underline{|n-1}}$.

7. Prove that $c_0c_2 + c_1c_3 + c_2c_4 + \ldots + c_{n-2}c_n = \dfrac{\underline{|2n}}{\underline{|n+2}\,\underline{|n-2}}$.

8. Prove that $c_0^2 - c_1^2 + c_2^2 - \ldots + (-1)^n c_n^2$ is equal to 0 if n be odd, and to $\dfrac{\underline{|n}}{\underline{|\tfrac{1}{2}n}\,\underline{|\tfrac{1}{2}n}}$ if n be even.

9. Prove that $c_0^2 + 2c_1^2 + 3c_2^2 + \ldots + (n+1)c_n^2 = \dfrac{(n+2)\,|\underline{2n-1}}{|\underline{n-1}\,|\underline{n}}$.

10. Prove that $c_1^2 + 2c_2^2 + 3c_3^2 + \ldots + nc_n^2 = \dfrac{|\underline{2n-1}}{|\underline{n-1}\;|\underline{n-1}}$.

11. Prove that $c_1 - \dfrac{c_2}{2} + \dfrac{c_3}{3} - \ldots + (-1)^{n-1}\dfrac{c_n}{n} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \dfrac{1}{n}$.

12. Prove that

$$\frac{1}{x} - \frac{c_1}{x+1} + \frac{c_2}{x+2} - \ldots + (-1)^n \frac{c_n}{x+n} = \frac{|\underline{n}}{x(x+1)(x+2)\ldots(x+n)}.$$

CHAPTER XXVII.

MISCELLANEOUS THEOREMS AND EXAMPLES.

257. IT is beyond the range of this book to shew how to find the factors of expressions of higher degree than the second when the expressions are in their most general forms; there are however many expressions of higher degree than the second whose factors can be seen by inspection, either at once or after some rearrangement, as in the following examples.

Ex. 1. Find the factors of $1 - x - x^3 + x^4$.

$$1 - x - x^3 + x^4 = 1 - x - x^3(1 - x) = (1 - x)(1 - x^3)$$
$$= (1 - x)(1 - x)(1 + x + x^2).$$

Ex. 2. Find the factors of $2x^3 - 3x^2 + 1$.

$$2x^3 - 3x^2 + 1 = 2x^2(x - 1) - (x^2 - 1)$$
$$= (x - 1)\{2x^2 - x - 1\} = (x - 1)(x - 1)(2x + 1).$$

Ex. 3. Find the factors of $(x^2 + 1)^2 - 2x(x^2 + 1) - 8x^2$.

$$(x^2 + 1)^2 - 2x(x^2 + 1) - 8x^2 = \{(x^2 + 1) + 2x\}\{(x^2 + 1) - 4x\}$$
$$= (x + 1)^2(x - 2 + \sqrt{3})(x - 2 - \sqrt{3}).$$

258. The following is an example of a quadratic expression which is somewhat more complex than any we have hitherto considered, but its factors can be found by the ordinary method.

Ex. Find the factors of $x^2 + 4xy + 3y^2 - 4x - 10y + 3$.

The given expression is

$$x^2 + 2x(2y - 2) + 3y^2 - 10y + 3.$$

The terms which contain x will be made a perfect square by the addition of $(2y - 2)^2$. We therefore add and subtract $(2y - 2)^2$: thus

$$x^2 + 2x(2y - 2) + (2y - 2)^2 - (2y - 2)^2 + 3y^2 - 10y + 3,$$

that is $$(x + 2y - 2)^2 - (y^2 + 2y + 1).$$

Now the last result is the difference of two squares; hence the factors are

$$x + 2y - 2 + (y + 1) \text{ and } x + 2y - 2 - (y + 1),$$

that is $$x + 3y - 1 \text{ and } x + y - 3,$$

and these are the factors required.

259. It is known that $x - a$, $x^2 - a^2$ and $x^3 - a^3$ are all divisible by $x - a$; we will now prove that $x^n - a^n$ is divisible by $x - a$ for all positive integral values of n.

$$\begin{aligned} x^n - a^n &= x^n - ax^{n-1} + ax^{n-1} - a^n \\ &= x^{n-1}(x - a) + a(x^{n-1} - a^{n-1}). \end{aligned}$$

Hence, *if* $x - a$ divides $x^{n-1} - a^{n-1}$ it will also divide

$$x^{n-1}(x - a) + a(x^{n-1} - a^{n-1}),$$

that is, it will divide $x^n - a^n$.

Thus, *if* $x - a$ divides $x^{n-1} - a^{n-1}$ it will also divide $x^n - a^n$.

But we know that $x - a$ divides $x^3 - a^3$; it will therefore also divide $x^4 - a^4$. And, since $x - a$ divides $x^4 - a^4$ it will also divide $x^5 - a^5$. And so on indefinitely.

Hence $x^n - a^n$ is divisible by $x - a$, when n is any positive integer.

Since $x^n + a^n = x^n - a^n + 2a^n$, it follows that when $x^n + a^n$

is divided by $x-a$ the remainder is $2a^n$, so that x^n+a^n is never divisible by $x-a$.

If we change a into $-a$, $x-a$ becomes $x-(-a)=x+a$; also x^n-a^n becomes $x^n-(-a)^n$, and $x^n-(-a)^n$ is x^n+a^n or x^n-a^n according as n is odd or even.

Hence, when n is *odd*

x^n+a^n is divisible by $x+a$,

and when n is *even*

x^n-a^n is divisible by $x+a$.

Thus, when n is a positive integer

$x-a$ divides x^n-a^n always,
$x-a$ .... x^n+a^n never,
$x+a$ .... x^n-a^n when n is even,
and $x+a$ .... x^n+a^n .... odd.

We have in the above shewn that the four cases are all included in the first: we leave it as an exercise for the student to prove each case separately.

The above results may be written so as to shew the quotients: thus

$$\frac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}a+x^{n-3}a^2+\dots+a^{n-1},$$

$$\frac{x^n\pm a^n}{x+a}=x^{n-1}-x^{n-2}a+x^{n-3}a^2-\dots\pm a^{n-1},$$

the upper or lower signs being taken on each side of the second formula according as n is odd or even.

EXAMPLES. LIX.

1. Find the factors of $x^3 - x^2 - x + 1$.

2. Find the factors of $x^6 - x^4 - x^2 + 1$.

3. Find the factors of $1 - x - x^4 + x^5$.

4. Find the factors of $1 - x^2 - x^8 + x^{10}$.

5. Find the factors of $x^2 - y^2 - 2ax - 2by + a^2 - b^2$.

6. Find the factors of $x^2 - y^2 - 3x - y + 2$.

7. Find the factors of $x^2 - 2xy - 8y^2 - 2x + 20y - 8$.

8. Find the factors of $a^2 - b^2 + c^2 - f^2 - 2ac - 2bf$.

9. Write down the quotient in each of the following divisions (i) $(x^5 - y^5) \div (x - y)$, (ii) $(x^7 + y^7) \div (x + y)$, (iii) $(x^8 - y^8) \div (x + y)$.

10. Shew that, if n is any positive integer, $7^n - 1$ is divisible by 6; shew also that $35^{2n+1} + 1$ is divisible by 36.

11. Shew without actual division that

$$(3x^2 - 2x + 1)^3 - (x^2 + 3x - 5)^3$$

is divisible by $x^2 - 5x + 6$.

12. *Write down* the result of dividing $(2a + 3b)^3 + (3a + 2b)^3$ by $5a + 5b$.

13. *Write down* the result of dividing $(2a + 4b - 4c)^3 + (a - b + 7c)^3$ by $a + b + c$.

14. Shew that $(1 - x)^2$ is a factor of $1 - x - x^5 + x^6$.

15. Shew that $(1 - x)^2$ is a factor of $1 - x - x^n + x^{n+1}$, n being any positive integer.

16. Shew that $(x - 1)^2$ is a factor of $nx^{n+1} - (n + 1)x^n + 1$, and also of $x^n - nx + n - 1$, n being any positive integer.

260. We now prove the following important proposition:

If any rational and integral expression which contains x vanish when a is put for x, then will x − a be a factor of the expression.

Let the expression, arranged according to powers of x, be

$$ax^n + bx^{n-1} + cx^{n-2} + \dots$$

Then, by supposition,

$$aa^n + ba^{n-1} + ca^{n-2} + \dots = 0.$$

Hence $$ax^n + bx^{n-1} + cx^{n-2} + \dots$$

$$= ax^n + bx^{n-1} + cx^{n-2} + \dots - (aa^n + ba^{n-1} + ca^{n-2} + \dots)$$

$$= a(x^n - a^n) + b(x^{n-1} - a^{n-1}) + c(x^{n-2} - a^{n-2}) + \dots$$

But, by the last Article, $x^n - a^n$, $x^{n-1} - a^{n-1}$, $x^{n-2} - a^{n-2}$, &c. are all divisible by $x - a$.

Hence also $ax^n + bx^{n-1} + cx^{n-2} + \dots$ is divisible by $x - a$.

261. The proposition proved in the preceding Article can also be proved in the following manner:

Divide the expression $ax^n + bx^{n-1} + cx^{n-2} + \dots$ by $x - a$, continuing the process until the remainder, if there be any remainder, does not contain x; and let Q be the quotient and R the remainder.

Then, by the nature of division,

$$ax^n + bx^{n-1} + cx^{n-2} + \dots = Q(x - a) + R,$$

and the two members of this equation are *identical.*

Now since R does not contain x, no change will be made in R by changing the value of x: put then $x = a$, and we have

$$aa^n + ba^{n-1} + ca^{n-2} + \dots = Q(a - a) + R = R.$$

Hence *if any expression which contains x be divided by*

$x-a$, the remainder is found by putting a in the place of x in the given expression.

This includes as a particular case the proposition that $x-a$ is a factor of any expression which vanishes when a is put for x.

Ex. 1. Find the remainder when x^3-4x^2+2x+1 is divided by $x-3$.

The remainder is $3^3-4.3^2+2.3+1=-2$.

Ex. 2. Shew that $x-2$ is a factor of x^4-3x^2+2x-8.

The condition is $2^4-3.2^2+2.2-8=0$; and this condition is satisfied.

Ex. 3. Find the L.C.M. of

$$x^2-3x+2, \text{ and } x^3-13x+12.$$

The factors of x^2-3x+2 are $x-1$ and $x-2$; and of these $x-1$ does, and $x-2$ does not satisfy the above condition of being a factor of $x^3-13x+12$. Thus the L.C.M. is $(x-2)(x^3-13x+12)$.

Ex. 4. Shew that any rational and integral expression is divisible by $x-1$ if the sum of the coefficients of the different powers of x is zero.

For if the sum of the coefficients is zero, the expression will vanish when we put $x=1$, and therefore $x-1$ is a factor.

262. We have proved that $x-a$ is a factor of the expression $ax^n+bx^{n-1}+cx^{n-2}+\ldots$, provided that the expression vanishes when a is put for x.

If the division were actually performed it is clear that the first term of the quotient, which is the term of the highest degree in x, would be ax^{n-1}.

Hence the given expression is equivalent to

$$(x-\alpha)(ax^{n-1}+\&c.).$$

Now suppose that the given expression also vanishes when $x=\beta$; then the product of $x-\alpha$ and $ax^{n-1}+\&c.$ will vanish when $x=\beta$; and, since $x-\alpha$ does not vanish when $x=\beta$, it follows that $ax^{n-1}+\&c.$ must vanish when $x=\beta$. Hence $x-\beta$ is a factor of $ax^{n-1}+\&c.$; and, if the division were performed, it is clear that the first term of the quotient would be ax^{n-2}.

Hence the original expression is equivalent to

$$(x-\alpha)(x-\beta)(ax^{n-2}+\&c.\ldots).$$

Similarly, if the original expression vanishes also for the values γ, δ, &c. of x, it is equivalent to

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\ldots(ax^{n-r}+\&c.),$$

where r is equal to the number of the factors $x-\alpha$, $x-\beta$, &c.

If therefore the given expression vanishes for n values α, β, γ, &c., there are n factors such as $x-\alpha$, and the remaining factor, $ax^{n-r}+\&c.$, will reduce to a; and hence the given expression is equivalent to

$$a(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$$

263. An expression of the n^{th} degree in x cannot vanish for more than n values of x; for, if the expression

$$ax^n+bx^{n-1}+cx^{n-2}+\ldots$$

vanish for the n values α, β, $\gamma\ldots$, it must be equivalent to

$$a(x-\alpha)(x-\beta)(x-\gamma)\ldots$$

If now we substitute any other value, P suppose, different from each of the n values α, β, γ, &c.; then, since no one of the factors $P-\alpha$, $P-\beta$, &c. is zero, their continued product cannot be zero, and therefore the given expression cannot vanish for the value $x = P$, *except a itself is zero.*

But, if a is zero, the original expression reduces to $bx^{n-1} + cx^{n-2} + \ldots$, and is of the $(n-1)^{\text{th}}$ degree; and hence it can only vanish for $n-1$ values of x, except b is zero. And so on.

Hence an expression of the n^{th} degree in x cannot vanish for more than n values of x, except the coefficients of all the powers of x are zero; and when all these coefficients are zero, the expression will clearly vanish for any value whatever of x.

264. The values of x for which the *expression*

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots$$

is equal to zero, are the roots of the *equation*

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots = 0.$$

Thus *an equation of the n^{th} degree cannot have more than n roots*, except the coefficients of all the different powers of the unknown quantity are zero, in which case any value of x satisfies the equation.

265. If the two expressions of the n^{th} degree

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots,$$

and
$$px^n + qx^{n-1} + rx^{n-2} + \ldots,$$

be equal to one another for more than n values of x, it follows that the equation

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots = px^n + qx^{n-1} + rx^{n-2} + \ldots,$$

that is, the equation

$$(a-p)x^n + (b-q)\,x^{n-1} + (c-r)x^{n-2} + \ldots = 0$$

has more than n roots.

Hence, by Art. 264, the coefficients of all the different powers of x must be equal to zero.

Thus $a=p,\quad b=q,\quad c=r,\quad$ &c.

Hence, if two expressions of the n^{th} degree in x are equal to one another for *more than n values of x*, the coefficient of any power of x in one expression is equal to the coefficient of the same power of x in the other expression.

266. When any two expressions, which have a finite number of terms, are equal to one another for *all values*, the condition of the last Article is clearly satisfied, for the number of values must be greater than the index of the highest power of any contained letter.

Hence when any two expressions, which have a finite number of terms, are equal to one another for *all values* of the letters involved in them, we may equate the coefficients of the different powers of any letter.

We may remark that the above proposition is true of two expressions which have an infinite number of terms, but the consideration of such expressions is beyond the range of this book.

267. An expression is said to be *symmetrical* with respect to any two letters when the expression is unaltered by interchanging the two letters.

Thus $a+b$ and a^2+b^2 are symmetrical with respect to a and b, for neither expression would be altered by changing a into b and b into a.

Also $a+b+c$ and $a^3+b^3+c^3-3abc$ are symmetrical with respect to any two of the three letters a, b, c.

It is clear that the only expression of the first degree which is symmetrical with respect to the three letters a, b, c is $pa+pb+pc$, where p is some numerical coefficient.

268. Although it is best for beginners to arrange factors which contain the three letters a, b, c so that a precedes b or c and that b precedes c, this arrangement is not the best for more advanced students.

The best arrangement is when a precedes b, b precedes c, and c precedes a. This is the arrangement we shall in future adopt for any three letters such as a, b and c.

Thus we write $bc + ca + ab$; and we put first the term which does not contain a, and then the term which does not contain b: an arrangement corresponding to this should always be observed.

269. We now proceed to consider some examples which will illustrate the theorems proved in the preceding articles.

Ex. 1. Shew that

$$(b-c)^5 + (c-a)^5 + (a-b)^5$$

is divisible by $(b-c)(c-a)(a-b)$.

If we put $b = c$ in the expression

$$(b-c)^5 + (c-a)^5 + (a-b)^5 \text{.................(i)},$$

the result is $(c-a)^5 + (a-c)^5$,

which is zero.

Hence $(b-c)$ is a factor of (i), and we can prove in a similar manner that $c-a$ and $a-b$ are factors.

Hence (i) is divisible by

$$(b-c)(c-a)(a-b).$$

Ex. 2. Shew that

$$(a+b+c)^3 - (b+c-a)^3 - (c+a-b)^3 - (a+b-c)^3 = 24abc.$$

If we put $a = 0$ in the expression

$$(a+b+c)^3 - (b+c-a)^3 - (c+a-b)^3 - (a+b-c)^3 \text{...(i)},$$

it is easy to see that the result will vanish.

Hence a is a factor of (i).

So also b and c are factors of (i).

Now (i) is an expression of the *third* degree; it can therefore only have *three* factors. Hence it is either equal to abc, or is equal to abc multiplied by some *number*.

Thus

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$$
$$= Labc \text{ ....(ii)},$$

where L is some number which is always the same whatever a, b and c may be.

We can find the value of L by giving particular values to a, b, and c.

Thus, let $a=b=c=1$. Then (ii) becomes

$$3^3-1^3-1^3-1^3=L;$$
$$\therefore \quad L=24.$$

We can also find the value of L by actually finding the coefficient of abc in the left member of (ii). By Art. 256 the coefficient of abc in $(a+b+c)^3$ is $\frac{\underline{|3}}{\underline{|1}\,\underline{|1}\,\underline{|1}}=6$, and similarly the coefficient of abc in each of the other cubes is -6: thus the whole coefficient is 24. The coefficient on the right is L; hence, as before, $L=24$.

Ex. 3. Find the factors of

$$a^3(b-c)+b^3(c-a)+c^3(a-b).$$

If we put $b=c$ in the expression

$$a^3(b-c)+b^3(c-a)+c^3(a-b),$$

the result is $\quad c^3(c-a)+c^3(a-c),$

which is clearly zero.

Hence $b-c$ is a factor of the given expression; and we can prove in a similar manner that $c-a$ and $a-b$ are also factors.

Now the given expression is of the *fourth* degree; hence, besides the three factors we have found, there must be one other factor of the *first* degree, and as this factor must be symmetrical in a, b, c it must be $a+b+c$.

Hence the given expression must be equal to

$$L(b-c)(c-a)(a-b)(a+b+c),$$

where L is a number.

We can find L by giving particular values to a, b, and c; or, by comparing the coefficients of a^3, we have at once $b-c=-L(b-c)$. Hence $L=-1$.

Thus

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c).$$

Ex. 4. Simplify

$$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}.$$

The L.C.M. of the denominators is $(b-c)(c-a)(a-b)$.

Hence the given expression is equal to

$$\frac{-a^2(b-c)-b^2(c-a)-c^2(a-b)}{(b-c)(c-a)(a-b)}.$$

Now we naturally test whether either of the factors of the denominator is also a factor of the numerator: we are thus led to find that the numerator is the same as the denominator, so that the given expression is equal to unity.

270. The following is an important identity:

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

It should be noticed that

$$a^2+b^2+c^2-bc-ca-ab=\tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$

Since $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$, it follows that $a^3+b^3+c^3-3abc=0$ if $a+b+c=0$.

Hence $a^3+b^3+c^3=3abc$ for all values of a, b and c, provided only that $a+b+c=0$.

That is, the sum of the cubes of any three quantities is equal to three times their product, provided that the sum of the three quantities is zero.

For example,

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b),$$

$$(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$$
$$=3(b+c-2a)(c+a-2b)(a+b-2c);$$

also, since

$$(x-a)(b-c)+(x-b)(c-a)+(x-c)(a-b)=0,$$

we have

$$(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$$
$$=3(x-a)(x-b)(x-c)(b-c)(c-a)(a-b).$$

271. We have seen that we must have more than one equation in order to determine the values of two or more unknown quantities. This however is by no means always the case when the values of the unknown quantities are in any way restricted.

For example, if we have the single equation $3x+4y=10$, and restrict both x and y to positive integral values, the equation can only be satisfied by one set of values, namely by the values $x=2$, $y=1$.

Again, from the relation

$$(x-a)^2+(y-b)^2=0,$$

with the restriction that all the quantities must be *real*, we can conclude both that $x=a$ and that $y=b$; for the squares of real quantities must be positive, and the sum of two positive quantities cannot be zero unless they are both zero.

Another example is considered in Art. 186.

Ex. Shew that, if a, b, c, d are all real and

$$(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd),$$

then will $$a=b=c=d.$$

We must write the expression

$$(a+b)^2+(b+c)^2+(c+d)^2-4(ab+bc+cd)$$

as the sum of two or more squares: we then have

$$(a-b)^2+(b-c)^2+(c-d)^2=0.$$

Then, since $a-b$, $b-c$, and $c-d$ are all real, we must have

$$a=b,\ b=c,\ \text{and}\ c=d.$$

272. The following are important examples.

Ex. 1. Find the greatest value of the product of two positive quantities whose sum is given.

Let the given sum be $2a$. Then, if one of the quantities be $a+x$, the other will be $a-x$; also the product will be a^2-x^2.

Now, since a is fixed, a^2-x^2 has its greatest value when $x=0$, and in that case the two quantities are equal to one another.

Thus, if the sum of two positive quantities is given, their product is greatest when they are equal to one another.

Ex. 2. Find the least value of the sum of two positive quantities whose product is given.

Since $(x+y)^2=4xy+(x-y)^2$, we see that $(x+y)^2$ is never less than $4xy$, and that $(x+y)^2$ is equal to $4xy$ when $x=y$.

Hence, if the product of two positive quantities is given, their sum is least when they are equal to one another.

Ex. 3. To find the least value of $x + \frac{a^2}{x}$.

Put $2s$ for this sum; then we have

$$x + \frac{a^2}{x} = 2s,$$

or $$x^2 - 2sx + a^2 = 0,$$

whence $$x = s \pm \sqrt{s^2 - a^2}.$$

Now x is to be *real;* therefore $\sqrt{s^2 - a^2}$ must be real, and therefore s cannot be less than a.

Hence the least value of $x + \frac{a^2}{x}$, for real and positive values of x, is $2a$.

Ex. 4. The arithmetic mean of any two positive quantities is greater than their geometric mean.

Let the two quantities be a and b.

The arithmetic mean is $\frac{1}{2}(a+b)$, and the geometric mean is $\sqrt{ab}$.

We have therefore to prove that

$$\tfrac{1}{2}(a+b) > \sqrt{ab},$$

or that $$(a+b)^2 > 4ab,$$

or that $$(a-b)^2 > 0,$$

which is the case, since $(a-b)^2$ is necessarily *positive.*

273. We shall conclude by shewing how to find the cube root of any algebraical expression.

We know that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Thus the cube of a binomial expression has *four* terms; also, when the cube is arranged according to powers of some letter, the cube roots of its extreme terms are the terms of the original binomial.

Conversely, if an expression which is a perfect cube contains four terms arranged according to powers of some letter, its cube root is the sum of the cube roots of the extreme terms.

Thus the cube root of any perfect cube which has only four terms can be written down by inspection.

For example, to find the cube root of

$$27a^6 - 54a^5b + 36a^4b^2 - 8a^3b^3.$$

The cube roots of the extreme terms are $3a^2$ and $-2ab$. Hence the required cube root is $3a^2 - 2ab$.

Again, the cube root of $a^9 - 6a^6b^2 + 12a^3b^4 - 8b^6$ is $a^3 - 2b^2$.

274. When an expression which contains only three different powers of a particular letter is arranged according to powers of that letter, there will only be four terms. For example, the expression

$$a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 6abc + 3b^2c + 3bc^2,$$

which contains only three different powers of a, namely a, a^2 and a^3, when arranged according to powers of a, is

$$a^3 + 3a^2(b+c) + 3a(b^2 + c^2 + 2bc) + (b^3 + c^3 + 3b^2c + 3bc^2),$$

that is $$a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3.$$

This expression therefore comes under the case considered in the previous Article; and its cube root is $a + b + c$.

275. We will now consider the most general case.

Suppose we have to find the cube root of $(A+B)^3$, where A stands for any number of terms of the root, and B for the rest; the terms in A and B being arranged according to descending (or ascending) powers of some letter, so that every term in A is of higher (or lower) degree in that letter than any term in B.

Also suppose that the terms in A are known, and that we have to find the terms in B.

Subtracting A^3 from $(A + B)^3$, we have the remainder

$$(3A^2 + 3AB + B^2)\,B.$$

Now from the mode of arrangement it follows that the term of the highest (or lowest) degree in the remainder is $3 \times$ square of the first term of $A \times$ first term of B.

Hence to obtain the *next term* of the required root, that is, to obtain the highest (or lowest) term of B, we *subtract from the whole expression the cube of that part of the root which is already found, and divide the first term of the remainder by three times the square of the first term of the root.*

This gives a method of finding the successive terms of the root after the first; and the first term of the root is clearly the cube root of the first term of the given expression.

We will exemplify the process of finding a cube root by obtaining the cube root of $x^6 - 6x^5y + 21x^4y^2 - 44x^3y^3 + 63x^2y^4 - 54xy^5 + 27y^6$.

$$\begin{aligned} & x^6 - 6x^5y + 21x^4y^2 - 44x^3y^3 + 63x^2y^4 - 54xy^5 \\ & \qquad + 27y^6. \\ (x^2)^3 &= \underline{x^6} \\ (x^2 - 2xy)^3 &= \underline{x^6 - 6x^5y + 12x^4y^2 - 8x^3y^3} \\ (x^2 - 2xy + 3y^2)^3 &= x^6 - 6x^5y + 21x^4y^2 - 44x^3y^3 + 63x^2y^4 - 54xy^5 \\ & \qquad + 27y^6. \end{aligned}$$

We first take the cube root of the first term of the given expression: we thus obtain x^2, the *first term* of the required root.

We then subtract the cube of x^2 from the given expression, and divide the first term of the remainder, namely $-6x^5y$, by $3 \times (x^2)^2$: we thus obtain $-2xy$, the *second term* of the root.

We then subtract the cube of $x^2 - 2xy$ from the given expression, and divide the first term of the remainder, namely $9x^4y^2$, by $3 \times (x^2)^2$: we thus obtain $3y^2$, the *third term* of the root.

Now subtract the cube of $x^2 - 2xy + 3y^2$ from the given expression and there is no remainder. Hence the given expression is $(x^2 - 2xy + 3y^2)^3$, so that the required root is $x^2 - 2xy + 3y^2$.

The cubes of x^2, $x^2 - 2xy$, &c. are placed under the given expression, like terms being placed in the same column: the remainder left after taking away any cube is then obvious.

It should be remarked that instead of finding each cube independently, some labour can be saved by making use of the previous cube; the saving of labour is not however sufficiently great to make this a matter of much importance.

276. The *fourth* root of an expression can be obtained by taking the square root of its square root; also the *sixth* root by taking the square root of its cube root.

There is no difficulty in finding any assigned root of an algebraical expression.

The following rule, which depends upon the Binomial Theorem, may be given for finding the nth root of any expression.

Arrange the expression according to descending powers of some letter, and take the nth root of the first term: this gives the *first term* of the required root.

Also, having found any number of terms of the root, subtract from the given expression the nth power of that part of the root which is already found, and divide the first term of the remainder by n times the $(n-1)$th power of the first term of the root: this gives the *next term* of the root.

Roots higher than the second are very rarely required except such as can easily be seen by inspection.

EXAMPLES. LX.

1. Shew that $a^2(b-c)+b^2(c-a)+c^2(a-b)$
$=bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b)$.

2. Shew that $a^4(b-c)+b^4(c-a)+c^4(a-b)$
$=-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.

3. Shew that $(b-c)^5+(c-a)^5+(a-b)^5$
$=5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$.

4. Shew that
$(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4=12abc(a+b+c)$.

5. Shew that $a(b-c)^3+b(c-a)^3+c(a-b)^3$
$=(b-c)(c-a)(a-b)(a+b+c)$.

6. Shew that $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$
$=-(b-c)(c-a)(a-b)(bc+ca+ab)$.

7. Shew that
$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)=-(b-c)(c-a)(a-b)(bc+ca+ab)$.

8. Shew that $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$
$=-(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$.

9. Find the factors of $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$.

10. Find the factors of $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$+(b+c-a)(c+a-b)(a+b-c)$.

11. Find the factors of
$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)-(b+c-a)(c+a-b)(a+b-c)$.

12. Find the factors of $a(b+c)(b^2+c^2-a^2)$
$+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2)$.

13. Shew that $(x+y+z)^{2n+1}-x^{2n+1}-y^{2n+1}-z^{2n+1}$ is divisible by $(y+z)(z+x)(x+y)$, and find the quotient when $n=1$, and when $n=2$.

14. Shew that $(x+y)^3-x^3-y^3=3xy(x+y)$, and that
$(x+y)^5-x^5-y^5=5xy(x+y)(x^2+xy+y^2)$.

15. Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

16. Simplify $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}$.

17. Simplify $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$.

18. Simplify $\dfrac{1}{a^2(a-b)(a-c)}+\dfrac{1}{b^2(b-c)(b-a)}+\dfrac{1}{c^2(c-a)(c-b)}$.

19. Simplify $\dfrac{b+c}{(a-b)(a-c)}+\dfrac{c+a}{(b-c)(b-a)}+\dfrac{a+b}{(c-a)(c-b)}$.

20. Simplify $\dfrac{b+c}{a(a-b)(a-c)}+\dfrac{c+a}{b(b-c)(b-a)}+\dfrac{a+b}{c(c-a)(c-b)}$.

21. Shew that $\dfrac{1}{(a-b)(a-c)(x+a)}+\dfrac{1}{(b-c)(b-a)(x+b)}$

$$+\frac{1}{(c-a)(c-b)(x+c)}=\frac{1}{(x+a)(x+b)(x+c)}.$$

22. Shew that $\dfrac{a}{(a-b)(a-c)(x+a)}+\dfrac{b}{(b-c)(b-a)(x+b)}$

$$+\frac{c}{(c-a)(c-b)(x+c)}=\frac{-x}{(x+a)(x+b)(x+c)}.$$

23. Shew that $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}$

$$+\frac{c^2}{(c-a)(c-b)(x+c)}=\frac{x^2}{(x+a)(x+b)(x+c)}.$$

24. Simplify $\dfrac{(b^2-c^2)^3+(c^2-a^2)^3+(a^2-b^2)^3}{(b-c)^3+(c-a)^3+(a-b)^3}$.

25. Prove the following:

(i) $a+b=\dfrac{a^2}{a-b}+\dfrac{b^2}{b-a}$.

(ii) $a+b+c=\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

(iii) $a+b+c+d=\dfrac{a^4}{(a-b)(a-c)(a-d)}+\dfrac{b^4}{(b-c)(b-d)(b-a)}$

$$+\frac{c^4}{(c-d)(c-a)(c-b)}+\frac{d^4}{(d-a)(d-b)(d-c)}.$$

26. Shew that $\dfrac{a^3}{(a-b)(a-c)(a-d)}+\dfrac{b^3}{(b-c)(b-d)(b-a)}$

$$+\frac{c^3}{(c-d)(c-a)(c-b)}+\frac{d^3}{(d-a)(d-b)(d-c)}=1.$$

27. Shew that $\dfrac{bcd}{(a-b)(a-c)(a-d)}+\dfrac{cda}{(b-c)(b-d)(b-a)}$

$$+\frac{dab}{(c-d)(c-a)(c-b)}+\frac{abc}{(d-a)(d-b)(d-c)}=1.$$

28. Shew that $\frac{bc}{(a-b)(a-c)}(x-a)^2 + \frac{ca}{(b-c)(b-a)}(x-b)^2$
$$+\frac{ab}{(c-a)(c-b)}(x-c)^2 = x^2.$$

29. Prove that $(b+c)^2+(c+a)^2+(a+b)^2-(c+a)(a+b)$
$-(a+b)(b+c)-(b+c)(c+a)=a^2+b^2+c^2-bc-ca-ab.$

30. Shew that $(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)$
$$=2(a^3+b^3+c^3-3abc).$$

31. Shew that $(b+c-a)^3+(c+a-b)^3$
$+(a+b-c)^3-3(b+c-a)(c+a-b)(a+b-c)=4(a^3+b^3+c^3-3abc).$

32. Shew that $(x^2+2yz)^3+(y^2+2zx)^3$
$+(z^2+2xy)^3-3(x^2+2yz)(y^2+2zx)(z^2+2xy)=(x^3+y^3+z^3-3xyz)^2.$

33. Shew that, if $x^2+y^2+z^2=(yz+zx+xy)$, then $x=y=z$.

34. Prove the following:

(i) if $2(a^2+b^2)=(a+b)^2$, then $a=b$;

(ii) if $3(a^2+b^2+c^2)=(a+b+c)^2$, then $a=b=c$;

(iii) if $4(a^2+b^2+c^2+d^2)=(a+b+c+d)^2$, then $a=b=c=d$;

and (iv) if $n(a^2+b^2+c^2+\ldots)=(a+b+c+\ldots)^2$, then $a=b=c=\ldots$, n being the number of the letters.

35. Shew that the least value of $x+\frac{1}{x}$ is 2, the least value of $x+\frac{4}{x}$ is 4, and the least value of $x+\frac{9}{x}$ is 6, x being real and positive.

36. Shew that $\frac{x^2-3x+4}{x^2+3x+4}$ cannot be greater than 7 nor less than $\frac{1}{7}$, for real values of x.

37. If the sum of a given number of positive quantities is fixed, their continued product will be greatest when they are all equal.

38. If the continued product of a given number of positive quantities is fixed, their sum will be least when they are all equal.

39. Prove that, if $\frac{yz-x^2}{y+z}=\frac{zx-y^2}{z+x}$, then will each fraction be equal to $\frac{xy-z^2}{x+y}$, and also equal to $x+y+z$.

40. If $\dfrac{ad-bc}{a-b-c+d}=\dfrac{ac-bd}{a-b-d+c}$, then will $a=b$, or $c=d$, or $a+b=c+d$.

41. Shew that, if x, y, z be determined by the equations:

$$(a-\alpha)^2x+(a-\beta)^2y+(a-\gamma)^2z=(a-\delta)^2,$$
$$(b-\alpha)^2x+(b-\beta)^2y+(b-\gamma)^2z=(b-\delta)^2,$$
$$(c-\alpha)^2x+(c-\beta)^2y+(c-\gamma)^2z=(c-\delta)^2;$$

then will $\quad (d-\alpha)^2x+(d-\beta)^2y+(d-\gamma)^2z=(d-\delta)^2,$

where d has any value whatever.

42. If the equation $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{c}{x+c}+\dfrac{d}{x+d}$ have a pair of equal roots, then either one of the quantities a or b is equal to one of the quantities c or d, or else $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}+\dfrac{1}{d}$. Prove also that the roots are then $-a$, $-a$, 0; $-b$, $-b$, 0; or 0, 0, $-\dfrac{2ab}{a+b}$.

43. If $2(x^2+x'^2-xx')(y^2+y'^2-yy')=x^2y^2+x'^2y'^2$, then will $x=x'$ and $y=y'$.

44. Shew that $a(a+d)(a+2d)(a+3d)+d^4$ is a perfect square.

45. Find the cube root of each of the following expressions:

(i) $x^3-24x^2y+192xy^2-512y^3$.

(ii) $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.

(iii) $x^6-9x^5+33x^4-63x^3+66x^2-36x+8$.

(iv) $8x^6-36x^5+102x^4-171x^3+204x^2-144x+64$.

46. Shew that

$$(y+z)(z+x)(x+y)+xyz=xyz(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

47. If a, b, c, x are all real quantities, and

$$(a^2+b^2)x^2-2b(a+c)x+b^2+c^2=0,$$

then a, b, c are in geometrical progression and x is their common ratio.

48. Shew that $(1+x)(1+x^2)(1+x^4)\ldots$ to n factors $=\dfrac{1-x^{2n}}{1-x}$.

49. If $s=a+b+c$, prove that

$$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$$

50. If $a+b+c+d=0$; then will

$$\begin{aligned}(a^3+b^3+c^3+d^3)^2&=9(bcd+cda+dab+abc)^2\\&=9(bc-ad)(ca-bd)(ab-cd)\\&=9(b+c)^2(c+a)^2(a+b)^2.\end{aligned}$$

EXAMINATION PAPERS.

I.

CAMBRIDGE LOCAL EXAMINATIONS. JUNIOR STUDENTS. 1885.

1. Simplify the expressions:

$$3a-2(b-c)-\{2(a-b)-3(c+a)\}-\{9c-4(c-a)\}$$

and
$$12\left(\frac{x+y}{2}-\frac{y-2x}{6}\right)-8\left(\frac{3y-x}{2}-\frac{3x+2y}{4}\right).$$

2. Find the product of

$$x^3+6x^2y+12xy^2+8y^3$$

and
$$x^3-3x^2y+3xy^2-y^3.$$

If the product of two expressions be $x^8+x^4y^4+y^8$, and one of them be x^2-xy+y^2, find the other.

3. Prove the identities:

$$(x+y+z)^3=(x+y-z)^3+(x-y+z)^3+(-x+y+z)^3+24xyz,$$

$$\frac{1}{\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)}+\frac{1}{\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right)}+\frac{1}{\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)}=1.$$

4. Find the square root of

$$(2x+1)(2x+3)(2x+5)(2x+7)+16.$$

5. State and prove the rule for finding the Least Common Multiple of two algebraical expressions, and find the L.C.M. of

$$x^{11}+x^4 \text{ and } x^{16}+x^7.$$

6. Solve the equations:

(1) $(3x-1)(4x+5)-(x-2)(2x+1)=(2x+3)(5x-2)+12.$

(2) $\frac{x-y}{3}-\frac{2y-3x}{6}=8,\qquad \frac{x}{6}-\frac{y}{3}=1.$

(3) $\frac{1}{x-2}+\frac{2}{x-1}=\frac{6}{x}.$

7. An egg-dealer bought a certain number of eggs at 1*s.* 4*d.* per score, and five times the number at 6*s.* 3*d.* per hundred. He sold the whole at 10*d.* per dozen, gaining £1. 7*s.* by the transaction. How many eggs did he buy?

8. The men in a regiment can be formed into a solid square, and also into a hollow square 4 deep, the number of men in the front in the latter formation exceeding the number of men in the front in the former formation by 25. Find the number of men in the regiment.

9. Solve the equations:

(1) $\dfrac{b}{x-a}+\dfrac{a}{x-b}-2=0.$ (2) $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{5}{2},\quad x^2+3y^2=28.$

10. Find the sum of a series of n terms in arithmetical progression. How many terms of the series 21, 18, 15, amount to 81?

If a, b, c be in geometrical progression, and x, y be the arithmetic means between a, b and b, c respectively, prove that

$$\frac{a}{x}+\frac{c}{y}=2.$$

11. Find the number of Combinations of m things taken r together.

Find the sum of all the integral numbers consisting of 5 figures, which can be formed by the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, no digit being used more than once in any number.

12. Enunciate the Binomial Theorem; find by means of it the 7th term of $(a-2x)^{12}$; and shew that the middle term of $(1+x)^{2n}$ is

$$\frac{1\,.\,3\,.\,5\ldots(2n-1)}{\underline{|n}}\,(2x)^n.$$

II.

CAMBRIDGE LOCAL EXAMINATIONS. SENIOR STUDENTS. 1885.

1. Solve the equations:

(i) $\dfrac{x^2+1}{x-1}+\dfrac{x^2-2}{x-2}=2x,$

(ii) $\dfrac{ax+b}{cx+b}+\dfrac{bx+a}{cx+a}=\dfrac{(a+b)(x+2)}{cx+a+b},$

(iii) $x\sqrt{x^2+12}+x\sqrt{x^2+6}=3,$

(iv) $3x^2+5xy=22,$
$11xy+3y^2=19.$

2. A number consists of two digits, one of which is treble the other. Another number is formed from the first by reversing the digits. The difference between the numbers is equal to 18. Find the numbers.

3. Shew that if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n}$, then each fraction will be equal to

$$\left\{\frac{\lambda_1 a_1^m+\lambda_2 a_2^m+\ldots+\lambda_n a_n^m}{\lambda_1 b_1^m+\lambda_2 b_2^m+\ldots+\lambda_n b_n^m}\right\}^{\frac{1}{m}}.$$

Shew that if $$\frac{x}{a+2b+c}=\frac{y}{2a+b-c}=\frac{z}{4a-4b+c},$$

then $$\frac{a}{x+2y+z}=\frac{b}{2x+y-z}=\frac{c}{4x-4y+z}.$$

4. Find the sum of n quantities which are in geometrical progression, and write down the formula for the sum of n quantities which are in arithmetical progression.

Employ the formulae to sum the following series, each to 6 terms:

(i) $1-3+9-27+\ldots$, (ii) $1-3-7-11-\ldots$.

The sum of the squares of the first n natural numbers is equal to $20n$. Find n.

5. Find the number of combinations of n things taken r at a time without assuming the formula for the number of permutations.

How many different sums can be made with the following coins:— a penny, a sixpence, a shilling, a half-crown, a crown, and a sovereign?

[Questions 6 and 7 were beyond the range of this book.]

III.

Cambridge Higher Local Examinations, 1881.

1. If $a=6$, $b=3$, $c=2$, find the value of $a^3-b^3-c^3-3abc$, and of

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

2. Investigate a rule for finding the Highest Common Divisor of two Algebraical Expressions.

Find that of x^4-11x^2+49 and $7x^4-40x^3+75x^2-40x+7$.

3. Reduce to its simplest form, the fraction

$$\frac{(a^2+2bc)^3+(b^2+2ca)^3+(c^2+2ab)^3-3(a^2+2bc)(b^2+2ca)(c^2+2ab)}{(a^2-bc)^3+(b^2-ca)^3+(c^2-ab)^3-3(a^2-bc)(b^2-ca)(c^2-ab)}.$$

4. Solve the equations,

$$\left.\begin{aligned} 7x+11y&=68 \\ 9x-\ \ 4y&=33 \end{aligned}\right\}, \qquad \begin{aligned} &2x^2-11x=63, \\ &\left.\begin{aligned} x^2-y^2&=8x-16, \\ 3xy&=6x+7y. \end{aligned}\right\} \end{aligned}$$

What two numbers are those whose sum is 47 and product 312?

5. Find the sum of a series of quantities in Geometrical Progression.

The sum of three quantities in Geometrical Progression is $\frac{7}{6}$, and the sum of their squares $\frac{133}{36}$. What are the quantities?

6. Find the number of combinations of n things taken r together.

If the number of combinations of n things taken r together be equal to the number of combinations of n things taken $2r$ together, and the number of combinations of n things taken $r+1$ together be equal to $\frac{11}{3}$ times the number of combinations of n things taken $r-1$ together, find n and r.

7. Prove the Binomial Theorem for a positive integral value of the index.

In the expansion of $(a+b)^{2n}$, prove that the sum of the squares of the first, third, &c., coefficients differs from the sum of the squares of the second, fourth, &c., by

$$\frac{2n\,(2n-1)\ldots(n+1)}{1\,.\,2\ldots n}.$$

IV.

PREVIOUS EXAMINATION. OCTOBER, 1885.

1. Define a *power*, an *index*, and a *term*.

Simplify $\quad a-2b-\{4a-(b-3c)+c\}$.

2. State the rule for the multiplication of any two algebraical expressions.

Multiply $x^2+9y^2+9z^2+3xy-3xz+9yz$ by $x-3y+3z$.

3. Divide $\quad 1+3a-24a^2+8a^4$ by $4a^2-6a-1$.

4. Prove the rule for the multiplication of two algebraical fractions.

Simplify: (i) $\dfrac{x^3-7x^2+12x-4}{x^4-5x^3+5x^2-15x+6}$.

(ii) $\dfrac{x^2-16y^2}{x^2-xy-12y^2}\times\dfrac{x^2-9y^2}{x^2+xy-12y^2}$.

5. Shew that any term may be transposed from one side of an equation to the other, provided its sign be changed.

Solve the equations:

(i) $\frac{1}{x+1}+\frac{1}{x+9}=\frac{1}{x+3}+\frac{1}{x+7}$.

(ii) $(b+c)x+(b-c)y=2ab$,
$(c+a)x+(c-a)y=2ac$.

6. Shew that the product of the two roots of the equation $x^2+ax+b=0$ is equal to b.

Solve the equations:

(i) $(a-b)x^2+(b-c)x+c-a=0$.

(ii) $\sqrt{2x+10}+2\sqrt{x+6}=2$.

Verify by substitution your solutions of each of these equations.

7. A man has 30 coins of which some are shillings and the rest are half-crowns. If he were to change the half-crowns for sixpences, and the shillings for pennies, he would have 234 coins. How many shillings has he?

8. Define *ratio* and *proportion*.

Find x, having given that

$$1 : 2 :: x-2 : x+3.$$

If $x : y :: a : b$, shew that

$$(x^2-a^2)(y^2-b^2)=(xy-ab)^2.$$

9. Having given the first term, and the common ratio of the successive terms of a geometrical progression; find the n^{th} term, and the sum of n terms.

Use the formulae to find the 5th term, and the sum of 5 terms, of the series 3, 5, $8\frac{1}{3}$, &c.

10. Find the sum of the series $1+1\frac{3}{4}+2\frac{1}{2}+\ldots$ to 12 terms; and the sum of the series $\frac{4}{3}+\frac{3}{4}+\frac{27}{64}+\ldots$ to infinity.

Find also the sum of all the even numbers between 51 and 191.

V.

CAMBRIDGE PREVIOUS EXAMINATION. DECEMBER, 1885.

1. Define an *algebraical expression*, and the *degree* of an expression. What is a *homogeneous* expression?

Find the value of $a^3+b^3+c^3-3abc$,
when $a=-3$, $b=-5$ and $c=8$.

2. Prove that the product of any number of quantities is the same in whatever order the factors are multiplied.

Find the continued product of

$$(x-a)^3,\ (x+a)^3 \text{ and } (x^2+a^2)^3.$$

3. Find the factors of (i) $16x^4-81$, (ii) a^3-a^2-a+1 and (iii) $5x^2-8xy+3y^2$.

4. Shew that the value of a fraction is unaltered by multiplying its numerator and denominator by the same quantity.

Simplify

$$\text{(i)}\ \frac{2-x}{1-2x}-\frac{2+x}{1+2x}-\frac{1-6x}{4x^2-1}. \qquad \text{(ii)}\ \frac{1}{x-3}-\frac{3}{x-1}+\frac{3}{x+1}-\frac{1}{x+3}.$$

5. Solve the equations:

$$\text{(i)}\quad \frac{1}{2}(x-2)-\frac{x-5}{9}+5\frac{x-1}{6}=0.$$

$$\text{(ii)}\quad y+\frac{3}{x}+1=0,\ y+\frac{5}{x}=9. \qquad \text{(iii)}\quad \frac{x}{a}+\frac{a}{x}=\frac{b}{a}+\frac{a}{b}.$$

6. The sum of the ages of a father and his son is 80 years; also one-twelfth of the product of their ages, in years, exceeds the father's age by 56. How old are they?

7. Prove, in the case when m, n, and p are positive integers, that $a^m \times a^n = a^{m+n}$, and that $(a^m b^n)^p = a^{mp} b^{np}$.

Divide $x+243y^{\frac{5}{3}}$ by $x^{\frac{1}{5}}+3y^{\frac{1}{3}}$.

8. Shew that any ratio is made more nearly equal to unity by adding the same quantity to each of its terms.

Find the least positive integer which must be added to each of the terms of 5 : 11 in order that the new ratio may be greater than $\frac{5}{9}$.

9. If $\qquad a : b :: ma+nc : mb+nd,$
shew that $\qquad c : d :: ma-nc : mb-nd.$

10. Shew how to insert any number of arithmetic means between two given quantities.

Insert 41 arithmetic means between 5 and -9; and find their sum.

ANSWERS TO THE EXAMPLES.

I. **1.** (i) 6, (ii) 59, (iii) 16.
2. (i) 9, (ii) 5, (iii) 9, (iv) 0, (v), 0, (vi) 11.
3. 16, 27, 64, 8, 4, 2.
4. (i) 7, (ii) $\frac{5}{2}$, (iii) 0, (iv) 0. **7.** 8, 8.
8. (i) 7, (ii) 256, (iii) 2, (iv) 24.
9. 98, 98, 98, 98; 56, 56, 56, 56.
10. 3, $5\sqrt{5}$, 7, $\frac{3}{2}$, 4.

II. **1.** (i) 1, (ii) -1, (iii) -7, (iv) -10.
2. (i) 1, (ii) 2, (iii) 0.
3. (i) -7, (ii) 7, (iii) 14. **4.** $-b+a$, 1.
5. -12. **6.** 120, -30. **7.** $-{\cdot}005$.
8. -5 min. **9.** -10 degrees. **10.** The latter.
11. The latter. **12.** 6, 0. **13.** 0, 6; 6, 0.
14. 4, 0. **15.** 0, 0.

III. **1.** $-8ab$. **2.** $12ab$. **3.** $-a^4$.
4. a^5. **5.** a^3b^3. **6.** $2a^3b$.
7. $-x^3y^3$. **8.** $-2x^4y^4$. **9.** $-12a^3b^3$.
10. $10x^5y^5$. **11.** $6a^7b^5c$. **12.** $8a^7c^3$.
13. $6a^4x^3y^3$. **14.** $-15a^4b^4c^4x^4y^4$.
15. a^2, $-a^3$, a^4. **16.** x^2, $-y^3$, z^4.
17. a^2b^2, $-b^3c^3$, x^4y^4. **19.** $9a^2b^4x^2$, $4x^4y^2$, $25a^2x^4$.
20. $-a^6b^3x^9$, $-27a^3x^6y^9$. **21.** a^2b^2, a^3b^6, $-a^4b^7c^3$.
22. 128. **23.** -49. **24.** 1.

IV. **1.** -2, $-\frac{5}{2}$, $\frac{2}{3}$. **2.** $-a$, $-4a$, $-\frac{3}{2}ab$.
3. $2xy^2$. $-\frac{4}{3}ab$, $\frac{3}{5}x^3y$. **4.** ad, $-abc^2d^2$.
5. $-2a^5b^4c^4$, $-4x^3y$, $-\frac{1}{3}xy$. **6.** $8abc$.

V. **1.** 17, 10. **2.** 6. **3.** 0, 0. **4.** $\frac{5}{8}a^2b^2c^3$.
5. $-3bx^2y^3$. **6.** (i) ± 4, (ii) 3, (iii) $\sqrt{5}$, (iv) 37, (v) -24.

VI. **1.** $2a$. **2.** 0. **3.** a. **4.** a^3-a.
5. $3a-a^3$. **6.** $2m^2-n^2$. **7.** $4a^2-4ab+4b^2$.
8. 0. **9.** 0. **10.** $-a-b-c$. **11.** $2x$.
12. $3a-10b+2$. **13.** $-x^2y-2xy^2-3y^3$. **14.** 0.

VII. **1.** $a+b$, $x+x^2$. **2.** $3b-3a$, $3-3x^2$. **3.** b, $2y$.
4. $-2x^2+6x-3$. **5.** $3x^2-3x+1$.
6. $3x^2-2xy+y^2$. **7.** $4x^3-x^2y-10xy^2+7y^3$.
8. $2a^3+5a^2b+4ab^2-3b^3$. **9.** $2a^2b^2+5b^4$.
10. $2ab+2bc-ca$. **11.** $-a^2-2b^2-c^2$.
12. $-4x^3+4x^2y-xy^2+y^3$.

VIII. **1.** $-y+z$. **2.** a. **3.** 3. **4.** 0. **5.** 1.
6. 0. **7.** $6x-y-5z$. **8.** $x-2y+5z+8$.
9. $4a-4c$. **10.** $6x-4y+3z$. **11.** $-2a-b+4c$.
12. $2a-4b-3c+d$.

IX. **1.** x^2-4y^2. **2.** $a^2-\frac{1}{4}b^2$. **3.** $1-a^3x^3$. **4.** $-y^3+b^3$.
5. x^3-9a^2x. **6.** y^3-4b^2y. **7.** $x^2-y^2+2yz-z^2$.
8. $16a^2-4b^2+12bc-9c^2$. **9.** a^4+a^2+1. **10.** a^4+1.
11. x^8+x^4+1. **12.** x^4-x^2+1.
13. $3x^4+5x^3y+8x^2y^2-8xy^3-8y^4$.
14. $-x^4+4x^3y-2x^2y^2-4xy^3-y^4$.

X. **1.** (i) $4a^2+12ab+9b^2$. (ii) $a^4-10a^3b+25a^2b^2$.
(iii) $9x^2y^2-6xy^3+y^4$. (iv) $a^2+4b^2+9c^2-4ab-12bc+6ac$.
(v) $a^4-2a^3b+3a^2b^2-2ab^3+b^4$. (vi) $x^4+2x^3+3x^2+2x+1$.
(vii) $x^4-2x^3y+3x^2y^2-2xy^3+y^4$.
(viii) $b^2c^2+c^2a^2+a^2b^2+2abc\,(a+b+c)$.
(ix) $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.
(x) $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$.
(xi) $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$.
(xii) $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$.

2. (i) $x^4+x^2y^2+y^4$. (ii) $9x^4-x^2y^2+4xy^3-4y^4$.
(iii) $-x^2+2xz+y^2-z^2$. (iv) $a^6+a^4b^2-a^2b^4-b^6$.

XI. **1.** $3a^4-5a^3b-12a^2b^2-ab^3+3b^4$. **2.** $-x^4+6x^2y^2-y^4$.

3. $\frac{9}{2}x^4+2x^2+\frac{8}{9}$. **4.** $a^3-3abc+b^3+c^3$.

5. $8a^3-18abc+27b^3+c^3$. **6.** a^4x^4-ax. **7.** a^4-b^4.

8. x^8-y^8. **9.** $x^8+x^4y^4+y^8$. **10.** a^8+a^4+1.

11. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$.

12. $-x^3+2x^2y+3x^2z+4xy^2-12xyz+9xz^2-8y^3+12y^2z$
$+18yz^2-27z^3$.

13. $x^8-2a^4x^4+a^8$. **14.** x^4+y^4. **15.** $a^4-a^2b^2+b^4$.

16. $8a^3+36a^2b+54ab^2+27b^3$, and $27a^3-54a^2b+36ab^2-8b^3$.

18. $a^3-8a^2+23a-26$.

XII. **1.** $\frac{5}{4}x^2-\frac{7}{4}xy+y^2$. **2.** $x+2y$. **3.** $x-\frac{1}{2}y$.

4. $1+ax+a^2x^2$. **5.** $-x^2-xy-y^2$.

6. $-3x-2$. **7.** $-x^2-3x-1$.

8. x^2-x+1. **9.** x^4-x^2+1.

10. x^2-2x+4. **11.** $-x-y+z$.

12. $-x-2y-z$. **13.** $-xy+xz+yz$.

14. $a^2b^2+abcd+c^2d^2$. **15.** $a^4x^4-a^3x^3+ax-1$.

16. $-a-2b+3c$. **17.** x^2-x+2.

18. x^2+x+2. **19.** $-x^5-2x^4-4x^3-5x^2-10x-20$.

20. $x^2+3ax+3a^2$. **21.** $1+x-x^2$.

22. $x^2+y^2+z^2-xy+xz+yz$.

23. $-4x^2-y^2-z^2-2xy+2xz-yz$.

24. x^4-3x^2+4x+1. **25.** $1+2x+3x^2+4x^3$.

26. $a^2+2ab-ac+b^2-bc+c^2$. **27.** $1+3x-3x^2$.

28. $5-3x+2x^2$. **29.** $x+y+z$.

30. $ax^2+(ac-b^2)x-b^2c$.

MISCELLANEOUS EXAMPLES. I.

A. **1.** 6. **2.** $3a^2+ab-4ac+2b^2+bc+c^2$.

4. $x^3+\frac{23}{4}x^2-\frac{4}{3}x+1$, $\frac{1}{16}x^4+\frac{1}{4}x^2y^2+y^4$. **5.** $12x^2+12$.

6. $2+4x+6x^2+8x^3+10x^4$.

B. **1.** 87, -5. **2.** $5b^4-3b^3a+3a^3b-5a^4$.

3. 0, $7x-7$, $2b-2a$. **4.** $9x^4-\frac{1}{9}$, $a^2+2ab+b^2-c^2$.

6. $x^2-(y+2)x+y^2+y+1$.

C. **1.** $-1\frac{1}{6}$. **2.** $2a^3+\frac{5}{2}a^2b+\frac{31}{6}b^3$.

3. $a^2x^4+(2a^2-1)x^2+a^2$, $4a^2+b^2+9c^2+4ab-12ac-6bc$.

4. $x^2+2xy+y^2$. **5.** $x^4-xy^3+y^4$.

D. **1.** 0, 0. **2.** $2x-12y+7z$. **3.** $-4ab-4ac$.
4. $a^3-2a^2b+2ab^2-b^3$, $1+x^8+x^{16}$.
5. x^2-xz+z^2, $(x+y)^2-(x+y)z+z^2$.

E. **1.** 8. **2.** $14x+2y$, $3x+15y$, $42x^2+216xy+30y^2$.
4. b^2-a^2. **5.** $5(b-a)$, $5(7a^2-11ab+7b^2)$.

F. **1.** $\frac{1}{4}$. **2.** $a^3-a^2(x+\frac{1}{2}y+\frac{1}{4}z)+a(\frac{1}{2}xy+\frac{1}{4}xz+\frac{1}{8}yz)-\frac{1}{8}xyz$.
3. 12. **4.** $x-y+z$, $2y^2$.

XIII. **1.** 3. **2.** 1. **3.** -4. **4.** -6.
5. 0. **6.** 0. **7.** $\frac{3}{4}$. **8.** $\frac{1}{13}$.
9. -2. **10.** -3. **11.** $5\frac{4}{9}$. **12.** 2.
13. -8. **14.** 6. **15.** -20. **16.** 5.
17. -5. **18.** $-2\frac{12}{25}$. **19.** 12. **20.** $-9\frac{5}{13}$.
21. $2\frac{1}{2}$. **22.** 13. **23.** 1. **24.** $-1\frac{3}{4}$.
25. $\frac{2}{3}$. **26.** $2\frac{1}{2}$. **27.** $-1\frac{1}{4}$. **28.** $-1\frac{1}{6}$.
29. $4\frac{1}{2}$. **30.** $-1\frac{3}{19}$. **31.** -5. **32.** -1.
33. 2. **34.** -16. **35.** 1. **36.** $a+b$.
37. $2a$. **38.** $b-a$. **39.** $\frac{a}{2}$. **40.** $\frac{b}{2}$.
41. a. **42.** ab. **43.** $-\frac{2ab}{a^2+b^2}$. **44.** 0.
45. $b-a$. **46.** $-\frac{a^2+b^2}{2a}$. **47.** $\frac{1}{2}(a+b)$.
48. $\frac{bc}{a}$. **49.** $-\frac{1}{2}(a+b)$. **50.** $\frac{1}{3}(a+b+c)$.

XIV. **1.** 99, 101. **2.** 18, 38. **3.** 10, 15.
4. $27\frac{1}{2}$, $72\frac{1}{2}$. **5.** 20. **6.** 21.
7. 20, 5. **8.** 15, 5. **9.** 12, 26.
10. 9, 22. **11.** 40. **12.** 420.
13. 14. **14.** 6. **15.** 420.
16. 112. **17.** 45, 55. **18.** 60, 40.
19. $13\frac{1}{3}$, $33\frac{1}{3}$. **20.** 24, 12. **21.** £20.
22. 25*s.*, 15*s.* **23.** £18. 15*s.* **24.** £150, £100.
25. £40, £35. **26.** £130, £125, £105.
27. £30, £15, £20. **28.** *A* £22. 10*s.*, *B* £27. 10*s.*, *C* £50.
29. £1. to each man, 10*s.* to each woman, 5*s.* to each child.
30. 1*s.* 8*d.* to each man, 8*d.* to each woman, 4*d.* to each child. **31.** In 5 years. **32.** 30 years ago.

33. 30, 10. **34.** 35, 5. **35.** *A* £25, *B* £35, *C* £40.
36. *A* $23\frac{1}{2}$, *B* $25\frac{1}{2}$, *C* $27\frac{1}{2}$, *D* $23\frac{1}{2}$. **37.** 10*d*. a lb.
38. 10*d*. a score. **39.** £12000. **40.** £1000, £500, £250.
41. 20 shillings, 4 half-sovereigns, and 4 sovereigns.
42. 4 sovereigns, 12 half-sovereigns, and 20 shillings.
43. 5 h. $27\frac{3}{11}$ min. **44.** 9 h. $32\frac{8}{11}$ min. **45.** 18 and 15.
46. 85 s., 35 w. **47.** 1500. **48.** 28. **49.** £270.
50. 30. **51.** £280. **52.** 3*s*. **53.** 15 days.
54. 8 days. **55.** 480.

XV. **1.** $x=1, y=-1\frac{1}{2}$. **2.** $x=3, y=2$. **3.** $y=9, x=100$.
4. $x=-2, y=-1$. **5.** $x=4, y=-3$. **6.** $x=1, y=-1$.
7. $x=-54\frac{3}{8}, y=1\frac{3}{8}$. **8.** $x=\frac{5}{6}, y=-\frac{5}{6}$. **9.** $x=-2, y=\frac{1}{2}$.
10. $x=1, y=1$. **11.** $x=5, y=-3$. **12.** $x=3\frac{2}{7}, y=6\frac{5}{7}$.
13. $x=5, y=2$. **14.** $x=5, y=-3$. **15.** $x=1, y=-1$.
16. $x=3, y=-1$. **17.** $x=y=-2$. **18.** $x=2\frac{1}{2}, y=1\frac{1}{2}$.
19. $x=5, y=2$. **20.** $x=\frac{207}{224}, y=\frac{425}{224}$. **21.** $x=8, y=16$.
22. $x=5, y=7$. **23.** $x=2\frac{4}{7}, y=2\frac{2}{3}$. **24.** $x=\frac{18}{69}, y=-19$.
25. $x=3, y=6$. **26.** $x=\frac{1}{2}, y=-\frac{3}{2}$.
27. $x=a+\dfrac{b}{2}, y=a-\dfrac{b}{2}$. **28.** $x=a, y=b$.
29. $x=2a+b, y=2b+a$. **30.** $x=\dfrac{a}{b}, y=\dfrac{b}{a}$.
31. $x=2b-a, y=2a-b$. **32.** $x=y=c$.
33. $x=8\frac{5}{7}, y=3\frac{1}{7}, z=3$. **34.** $x=-1, y=-2, z=4$.
35. $x=\frac{1}{2}(b+c), y=\frac{1}{2}(c+a), z=\frac{1}{2}(a+b)$.

XVI. **1.** 20, 30. **3.** £$12\frac{1}{2}$, £$2\frac{1}{2}$. **4.** $\frac{2}{3}$ tons, $\frac{3}{2}$ tons.
5. *A*, 40 days, *B*, 120 days. **6.** $\frac{3}{5}$. **7.** $\frac{2}{3}$.
8. 240, 360. **9.** £22, £26. **10.** 5 shillings.
11. 480 yards. **12.** 24 days. **13.** £2400, £900.
15. 12, 24, 36, 48. **16.** *A*, £450, *B*, £225, *C*, £$237\frac{1}{2}$, *D*, £$87\frac{1}{2}$.
17. 48. **18.** 7 and 2. **19.** $\frac{31}{45}$. **20.** 1*s*. 6*d*.
21: 200, 250. **22.** 3 half-crowns, 8 shillings, 9 sixpences.
23. $17\frac{1}{2}$ francs. **24.** 3*s*. 6*d*., 4*s*. 2*d*. **25.** 50.

MISCELLANEOUS EXAMPLES. II.

A. **1.** 1. **2.** $7a-7b$. **4.** x^2-2x-2.
5. (i) 2, (ii) $x=6, y=15$, (iii) $x=\frac{1}{2}(a-b)$, $y=\frac{1}{2}(a+b)$.
6. 60.

B. **1.** $4\frac{1}{4}$. **2.** a^2+b^2. **4.** x^4-2x^2+x+2.
5. (i) -6; (ii) $x=-162$, $y=42$; (iii) $x=6$, $y=8$.
6. 90 sovereigns, 40 crowns.

C. **1.** $-b-d$.
4. $a^2+b^2+c^2-bc-ca-ab$, $x^2+y^2+1+y+x-xy$.
5. (i) $x=-1$, $y=-1$, (ii) $x=y=a$.
6. *A*, 12 hits and 18 misses. *B*, 24 hits and 6 misses.

D. **1.** $-11\frac{1}{4}$. **2.** $2ab$. **4.** $15b^2c^3-3ab^3c$, $a+2b-3c$.
5. (i) $\frac{7}{5}$; (ii) $x=5$, $y=4$; (iii) $x=\frac{1}{4}$, $y=\frac{1}{2}$. **6.** 1225.

E. **1.** a^2. **2.** $4a^2+ab+ac+4b^2+bc+4c^2$.
4. $x^2-\frac{1}{2}x+\frac{2}{3}$, cx^2+dx-c.
5. (i) 5; (ii) $x=\frac{3}{10}$, $y=\frac{3}{2}$; (iii) $x=a+b$, $y=a-b$.
6. 8 and 5.

F. **1.** 130. **2.** a^8-x^8. **4.** $2y^2-3y+1$, $a+2b+3c$.
5. (i) $\dfrac{cd-ab}{a+b-c-d}$; (ii) $x=-1$, $y=-1$. **6.** Tea 2*s.* 6*d.*, coffee 1*s.* 8*d.*

XVII. **1.** $(3a-b)^2$. **2.** $(2x-3)^2$.
3. $\left(x+\dfrac{y}{2}\right)^2$. **4.** $\left(2x^2-\dfrac{y}{4}\right)^2$.
5. $(2ax+by)^2$. **6.** $3(1+a^2x)^2$.
7. $-(a-2b)^2$. **8.** $-4(1-x)^2$.
9. $y^2\left(2x+\dfrac{y}{2}\right)^2$. **10.** $xy(x-2y)^2$.
11. $(2xy+a+b)^2$. **12.** $(3a+3b-c)^2$.

XVIII. **1.** $(a^2+b)(a^2-b)$. **2.** $(x+3y^2)(x-3y^2)$.
3. $(a+\sqrt{2})(a-\sqrt{2})$. **4.** $(x+\sqrt{3y})(x-\sqrt{3y})$.
5. $(a^2+4b^2)(a^2-4b^2)$. **6.** $(9x^2y^2+a^2)(9x^2y^2-a^2)$.
7. $xy(x+2y)(x-2y)$. **8.** $y(2x+3y^2)(2x-3y^2)$.
9. $3x^3(x+3a)(x-3a)$. **10.** $5ax(2ax+3y)(2ax-3y)$.
11. $4(a-b+c)(2c-b)$. **12.** $8x(x+1)^2(x-1)$.

XIX. **1.** $4(a+2b)(a^2-2ab+4b^2)$.
2. $(3xy-\frac{1}{2}z)(9x^2y^2+\frac{3}{2}xyz+\frac{1}{4}z^2)$.
3. $a^3(1-2a)(1+2a+4a^2)$. **4.** $(2a^2+b)(4a^4-2a^2b+b^2)$.
5. $2x(y-\frac{1}{2}x)(y^2+\frac{1}{2}xy+\frac{1}{4}x^2)$.
6. $9ab^2(a-\frac{1}{3}b)(a^2+\frac{1}{3}ab+\frac{1}{9}b^2)$.

7. $3ab\,(a+2b)\,(a^2-2ab+4b^2)$.

8. $5bc\,(2a-bc)\,(4a^2+2abc+b^2c^2)$.

9. $(a^3+8)\,(a^3-8)=(a+2)\,(a-2)\,(a^2+2a+4)\,(a^2-2a+4)$.

10. $(x^6+a^3b^3)\,(x^6-a^3b^3)=(x^2+ab)\,(x^2-ab)$
$(x^4+x^2ab+a^2b^2)\,(x^4-x^2ab+a^2b^2)$.

11. $9\,(x+y)\,(x^2+xy+y^2)$. **12.** $4\,(x-y)\,(7x^2-2xy+7y^2)$.

XX. **1.** $(x-1)\,(x-3)$. **2.** $(x+4)\,(x+5)$.
3. $(x-7)\,(x-2)$. **4.** $(x-12)\,(x-11)$.
5. $(x+7)\,(x-5)$. **6.** $(x-5)\,(x+2)$.
7. $(x+7)\,(x-2)$. **8.** $(x-12)\,(x+11)$.
9. $(x+2y)^2$. **10.** $(x-y)\,(x-4y)$.
11. $(4x-y^2)\,(x-y^2)$. **12.** $(5x-1)\,(2x+1)$.
13. $-(x-10)\,(x+1)$. **14.** $-(x^2-4)\,(x^2+2)$.
15. $x\,(x-6)\,(x+3)$. **16.** $xy\,(x-2y)\,(x+y)$.

XXI. **1.** $(4x^2+25y^2)\,(2x+5y)\,(2x-5y)$.
2. $(9a^2+4b^2)\,(3a+2b)\,(3a-2b)$.
3. $(1+3a)\,(1-3a+9a^2)$. **4.** $(3+2x)\,(9-6x+4x^2)$.
5. $2b\,(a-b)$. **6.** $(a+b+2c-2d)\,(a+b-2c+2d)$.
7. $(a-c)\,(a+b+c)$. **8.** $(a^2+c^2+b^2)\,(a+c)\,(a-c)$.
9. $(2a+b+c)\,(b-c)$. **10.** $8xy\,(x^2+y^2)$.
11. $(a+b+3c)\,(a+b-c)$. **12.** $(a-b+c)\,(a-b-c)$.
13. $ab\,(2a+b)\,(a+2b)$. **14.** $5\,(x+y)\,(7x^2+11xy+7y^2)$.
15. $(x-y)\,(x+\frac{1}{4}y)$. **16.** $(x+a)\left(x+\dfrac{1}{a}\right)$.
17. $(x+\frac{2}{3})\,(x+\frac{3}{2})$. **18.** $(x+\frac{2}{3})\,(x-\frac{3}{2})$.
19. $(3x+2y)\,(2x-3y)$. **20.** $(9x+8)\,(8x-9)$.
21. $(x^2-1)\,(x^2-4)=(x+1)\,(x-1)\,(x+2)\,(x-2)$.
22. $(x^2-9y^2)\,(x^2-4y^2)=(x+3y)\,(x-3y)\,(x+2y)\,(x-2y)$.
23. $(mx+ny)\,(nx+my)$. **24.** $(2x+13)\,(x-3)$.
25. $(a+b-3x)\,(a+b-2x)$.
26. $\{x+y-3\,(a+b)\}\,\{x+y-5\,(a+b)\}$.
27. $2\,(a-d)\,(a+b+c+d)$.
28. $(x^2+4x+3)\,(x^2+4x-5)=(x+1)\,(x+3)\,(x-5)\,(x-1)$.
29. $(a-c+b-d)\,(a-c-b+d)$.
30. $(2ab+a^2+b^2-c^2)\,(2ab-a^2-b^2+c^2)$
$=(a+b+c)\,(a+b-c)\,(c-a+b)\,(c+a-b)$.
31. $(3a+b)^2-4c^2=(3a+b+2c)\,(3a+b-2c)$.
32. $(a-y)\,(a^3+z^3)=(a-y)\,(a+z)\,(a^2-az+z^2)$.

XXII. **1.** a^2b^2. **2.** abc^2. **3.** $3ab$. **4.** $2xy$.
5. $12a^2b^3x^4$. **6.** b^3x. **7.** x^3y^5. **8.** b^3c^6.
9. $7y^2$. **10.** ab. **11.** xyz^2. **12.** $abcx^2$.

XXIII. **1.** $x^2(x-a)^3$. **2.** $a+b$. **3.** $(a+b)^2$.
4. $b^4c^3(b+c)^2$. **5.** $a(a^2+b^2)$. **6.** a^2+3b^2.
7. $a^2x^2(x+2a)$. **8.** $a^2x^2(a^2-4x^2)$. **9.** $x+2$.
10. $x(x+2y)$. **11.** $x-1$. **12.** $a-b$.

XXIV. **1.** $x-1$. **2.** $x-y$. **3.** $2x-1$.
4. $2x-y$. **5.** x^2-2. **6.** x^2-2y.
7. $a-2$. **8.** $2b-1$. **9.** $x-1$.
10. x^2y^2-1. **11.** $x-2a$. **12.** $2a-b$.
13. a^2-ab+b^2. **14.** $4a^2-2a+1$. **15.** x^2-y^2.
16. $x-1$. **17.** x^2+x+1. **18.** x^2-3x+5.
19. $x-7$. **20.** $x-y$. **21.** $x-y$.
22. $x-5$. **23.** x^2-2x+4. **24.** $a-2x+3$.
25. x^2+7. **26.** x^2+5x+1. **27.** x^2+4x+3.
28. x^2+3x+1. **29.** $y-2$. **30.** $y^2-3yz+4z^2$.
31. $2x^2-3x-1$. **32.** x^2+2x+2. **33.** $x-1$.
34. x^2-3x+1. **35.** $x^2+(2m-3)x-6m$.
36. $mx+ny$.

XXV. **1.** a^3b^3. **2.** a^2bc^3. **3.** $18a^2b^3$.
4. $20x^3y^3$. **5.** $120a^3b^4x^6$. **6.** $3a^2b^5x^5$.
7. $72a^2b^3x^4y^6$. **8.** $6a^3b^3c^3$.
9. $462ab^3xy^4z^3$. **10.** $a^2b^3c^2$.
11. $30x^2y^3z^3$. **12.** $a^4b^2c^3x^4$.

XXVI. **1.** $(a-x)(a-2x)(a-3x)$.
2. $a^3x^3(a-x)(a-2x)(a-3x)$.
3. $(a-b)(a+b)^2$. **4.** $12a^2b(a-b)(a+b)^2$.
5. $(x+1)(x+2)(x+4)$. **6.** $(x+y)(x+2y)(x+4y)$.
7. $(x-1)(x-2)(x-3)$. **8.** $(3x-y^2)(2x-y^2)(x-y^2)$.
9. $(a^2-b^2)^2$. **10.** $(x^2-4y^2)^2$.
11. $(x+2)(x+3)(x+4)$. **12.** $(x-2y)(x-3y)(x-4y)$.

XXVII. **1.** $54a^3b^2c^2$. **2.** $a^4b^3c^3$. **3.** $24a^3b^2$.
4. $12a^3b^3$. **5.** $x^3y^3(x^2-y^2)^2$. **6.** $a^2(a^2-b^2)$.
7. $xy^2(x^2-1)$. **8.** $12ax^2y^2(x-y)(x+y)^2$.
9. $90(x^2-9)(x^2-16)$. **10.** $12(a^2-b^2)(a^2-4b^2)$.

11. $8ab^3d\,(a^2-d^2)$. **12.** $(x-2)(x-4)(x-6)$.
13. $(x+1)(x+3)(x-10)$. **14.** $6\,(x^2-1)(x^2-4)$.
15. $(x-1)(x-2)(2x+3)(3x+1)$.
16. $(2x-1)(3x+2)(7x+1)$.
17. $(x-3y)(x-y)(3x-y)$. **18.** $x(x-1)(x+3)(2x-1)$.
19. $xy^2(x^2-4)(x^2-9)$. **20.** x^6-a^6.
21. x^4-1. **22.** $(x-4)(3x-2)(3x^2+2x+1)$.
23. $(x^2-a^2)^2$. **24.** $(x+1)(x+3)(x^2-4)$.
25. $(x-1)(x+2)(x+6)(x^2-2x+4)$.
26. $(a+1)(a+2)(a+3)(a+4)(a+5)$.

XXVIII. **1.** $\frac{a}{b}$. **2.** $\frac{x}{y}$. **3.** $\frac{c}{2a^2b^2}$.
4. $\frac{3}{2a^2b^2}$. **5.** $\frac{y^4}{x^2}$. **6.** $\frac{3}{4}x^5z^8$.
7. $\frac{3ac^4}{5b^2x^3}$. **8.** $\frac{5cd^3}{6a^3b}$. **9.** $\frac{3a^4x^2z^3}{5b^4y^2}$.
10. $\frac{5a^3c^2}{7y}$. **11.** $\frac{2c^4z^4}{3a^4x^4}$. **12.** $\frac{3ac^3x^2}{4b^2y^2z^2}$.

XXIX. **1.** $\frac{2b}{a+b}$. **2.** $\frac{y^2}{1-y^2}$. **3.** $\frac{a-b}{a+b}$.
4. $\frac{x}{x-a}$. **5.** $\frac{x-1}{x+1}$. **6.** $\frac{1-y}{1+y}$.
7. $\frac{x}{x-2}$. **8.** $\frac{x^2}{x^2-1}$. **9.** $\frac{4}{x+4}$.
10. $\frac{2x}{1+2x}$. **11.** $-\frac{1}{x+2}$. **12.** $-\frac{1}{a+3}$.
13. $-\frac{x^2}{x^2+a^2}$. **14.** $-\frac{x^2+a}{a}$. **15.** $\frac{x^3}{x^2+a^2}$.
16. $-\frac{x^2y^2}{x^2y^2+a^2}$. **17.** $\frac{5a}{x+3a}$. **18.** $-\frac{x}{x+4a}$.
19. $\frac{a-x}{a+x}$. **20.** $\frac{a^2+b^2}{a^2-b^2}$. **21.** $\frac{x+1}{x^2+x+1}$.
22. $\frac{x^2+1}{x^4+x^2+1}$. **23.** $\frac{x-2}{x-4}$. **24.** $\frac{1-2a}{1-4a}$.
25. $\frac{x-4}{x+11}$. **26.** $\frac{1-4y^2}{1+11y^2}$. **27.** $\frac{x-y}{x+4y}$.

28. $\dfrac{1-a^2b^2}{1+4a^2b^2}$. **29.** $\dfrac{a^2+ax+x^2}{a^2-ax+x^2}$. **30.** $\dfrac{a-b}{a+b}$.

31. $\dfrac{a^2+b^2}{a^2+ab+b^2}$. **32.** $\dfrac{a^2-ab+b^2}{a^2+b^2}$.

XXX. **1.** $\dfrac{x+2}{2x+1}$. **2.** $\dfrac{3x-5}{x^2-3x+2}$. **3.** $\dfrac{x^2+5x+10}{x^3+2x^2+3x+6}$.

4. $\dfrac{x+1}{x}$. **5.** $\dfrac{2x^2+3ax+7a^2}{x^2-6ax+2a^2}$.

6. $\dfrac{x^2+2x+3}{3x^2+2x+1}$. **7.** $\dfrac{x-2}{x^2+1}$.

8. $\dfrac{x^2-2x+1}{x^2-3x+1}$. **9.** $\dfrac{x+2}{2x+1}$.

XXXI. **1.** $\dfrac{2b}{a^2-b^2}$. **2.** $\dfrac{1}{x^2-5x+6}$. **3.** $\dfrac{x}{(x-2)^2}$.

4. $\dfrac{4y}{x-y}$. **5.** $\dfrac{a^2+b^2}{a^2-b^2}$. **6.** $\dfrac{1}{a}$.

7. $\dfrac{8(x+6)}{x^2-16}$. **8.** $\dfrac{x+3}{x^4-1}$. **9.** $\dfrac{1}{1-9x^2}$.

10. $\dfrac{4a^4}{a^4-x^4}$. **11.** $\dfrac{8a^8}{a^8-x^8}$.

12. $\dfrac{2(x^2+5x+7)}{(x+1)(x+2)(x+3)(x+4)}$. **13.** $\dfrac{2}{x(x^2-1)}$.

14. $\dfrac{2}{a(a+1)(a+2)}$. **15.** $\dfrac{48}{(x^2-9)(x^2-1)}$.

16. $\dfrac{6}{a(a+1)(a+2)(a+3)}$. **17.** $\dfrac{24}{x(x^2-1)(x^2-4)}$.

18. $\dfrac{24}{a(a+1)(a+2)(a+3)(a+4)}$. **19.** 0. **20.** 0.

21. $\dfrac{2}{(x-1)(x-2)(x-3)}$. **22.** 0. **23.** $\dfrac{x}{(x-2a)^2}$.

24. $\dfrac{1}{a+2}$. **25.** $\dfrac{2(x+ay)}{x-ay}$. **26.** $\dfrac{4x^2+3}{x^2-1}$.

27. $\dfrac{3-x}{1-x}$. **28.** $\dfrac{2}{x-3y}$. **29.** 1. **30** 1.

XXXII. **1.** $\frac{1}{2}$. **2.** $\frac{ab}{4c^2}$. **3.** $\frac{ad}{c^2}$.

4. $\frac{2ac}{3b^2}$. **5.** 1. **6.** 1. **7.** $\frac{b^2}{c^2}$.

8. $\frac{a^4}{c^4}$. **9.** $\frac{8}{abc}$. **10.** xyz. **11.** x^2.

12. 1. **13.** $\frac{2xy}{3ab}$. **14.** $\frac{xy}{b^2z}$.

15. $\frac{x^2+ax+a^2}{x-2a}$. **16.** $\frac{a^2+ax+x^2}{a^2-ax+x^2}$.

17. $\frac{x-1}{x-4}$. **18.** $\frac{1}{(x+1)(x+2)(x+3)}$.

19. 1. **20.** $\frac{x^2-6x+5}{x^2-6x+8}$. **21.** 1.

XXXIII. **1.** $\frac{5(x+a)}{(x-2a)^2}$. **2.** $\frac{1}{2x^2-1}$. **3.** 1. **4.** 0.

5. 0. **6.** $\frac{2}{y}$. **7.** 1. **8.** $x-1$.

9. $\frac{3(4x+5)(3x+4)}{(2x+3)(5x+6)}$. **10.** $\frac{1+x^4}{x(1+x^2)}$. **11.** $\frac{1}{3x^3}$.

12. $\frac{x^2(x-1)}{(x+1)}$. **13.** $\frac{4}{4x-a+3}$. **14.** 1.

XXXIV. **1.** 7. **2.** 4. **3.** 4. **4.** $-\frac{1}{5}$.

5. -2. **6.** -4. **7.** $-\frac{8}{7}$. **8.** $\frac{13}{15}$.

9. 1. **10.** -2. **11.** $\frac{3}{2}$. **12.** 5.

13. $\frac{1}{16}$. **14.** -2. **15.** 1. **16.** $6\frac{1}{5}$.

17. $\frac{38}{41}$. **18.** -6. **19.** $\frac{3}{2}$. **20.** -3.

21. $2a-b$. **22.** $\frac{a+b}{2}$. **23.** $\frac{-2bc}{a^2+b^2}$.

24. $\frac{ab}{b-a}$. **25.** $\frac{-2ab}{a+b}$, or 0.

26. $x=\frac{ab(a+b)}{a^2+b^2}$, $y=\frac{ab(a-b)}{a^2+b^2}$.

27. $x=2$, $y=2$.

MISCELLANEOUS EXAMPLES. III.

A. **1.** $y^3-6xy^2+6x^2y-x^3$.

3. (i) $(x^2+y^2)(x+y)(x-y)$, (ii) $(a-b)(a+b+2)$,
(iii) $(x-7)(x+4)$.

4. x^2+x-4, $\dfrac{x^2-x+5}{x(5x^2+4x+16)}$.

5. (i) 5, (ii) $x=1$, $y=-1$, (iii) $x=-\dfrac{a+b}{2}$.

6. *A* has £11. 15*s*., *B* has £3. 5*s*.

B. **1.** 0, 0.

3. (i) $x(x-2)$, (ii) $(2x-1)(x-2)$, (iii) $(x^2+1)(x-1)$.

4. $\dfrac{2}{x+3}$. **5.** (i) $\frac{3}{2}$, (ii) $x=a$, $y=0$. **6.** $\frac{3}{50}$.

C. **3.** (i) $(1-3x)(1+21x)$, (ii) $3y(x-2y)(x^2+2xy+4y^2)$,
(iii) $(x+1)^3(x-1)^3$.

4. $\dfrac{x-3}{x^2+7x+3}$, $\dfrac{x}{(x-1)(x-2)(x-3)}$.

5. (i) 5, (ii) $x=\dfrac{c(b-c)}{a(b-a)}$, $y=\dfrac{c(a-c)}{b(a-b)}$.

b

6. *A*, £70; *B*, £30.

D. **1.** 4. **3.** (i) $x(x+3y)(x-2y)$, (ii) $(x-a)(x+a)^2$,
(iii) $(x-1)(x+1)(y-1)(y+1)$.

4. (i) $\dfrac{2(a+x)}{a-x}$, (ii) 2. **5.** (i) $x=5$, (ii) $x=\frac{1}{5}$, $y=-17$.

6. 72.

E. **2.** $x^3+x^2y+xy^2+y^3$, $(x+y)^3+2z(x+y)^2+4z^2(x+y)+8z^3$.

3. $(x+5)(5x-1)$, $a(a+b+c)(a-b-c)$.

4. $2a(2a-3b)(2a+3b)(2a-b)$. **6.** £240.

F. **1.** 59. **2.** $x^5-ax^4-a^4x+a^5$.

3. $(x^2+1)(x+1)^2(x-1)^2(x^2+x+1)(x^2-x+1)$.

4. $\dfrac{x}{4x^2-y^2}$. **5.** (i) $-\dfrac{a^2+b^2}{a+b}$, (ii) $x=3$, $y=-1$, $z=0$.

6. 350 acres.

XXXV. **1.** $\pm\frac{1}{3}$. **2.** -1 or -2. **3.** 0 or $\pm\frac{1}{2}$.
4. ± 1 or ± 2. **5.** ± 1. **6.** $\pm\sqrt{15}$.
7. ± 4. **8.** 0. **9.** 0 or -7.
10. 0 or 11. **11.** 0 or $1\frac{1}{3}$. **12.** 0 or $1\frac{1}{5}$.
13. 0 or $\frac{b}{a}$. **14.** 0 or $-c$. **15.** ± 3.
16. ± 3. **17.** ± 1. **18.** ± 9.
19. 1 or 2. **20.** 2 or 5. **21.** 4 or -2.
22. 2 or -3. **23.** $0, 4$ or -2. **24.** $0, 2$ or -3.
25. $-a, -b,$ or $-c$. **26.** $-b-c, -c-a,$ or $-a-b$.

XXXVI. **1.** ± 2. **2.** ± 3. **3.** 1 or 6.
4. 3 or 6. **5.** 6 or -3. **6.** $-1\pm\sqrt{2}$.
7. 3 or -1. **8.** 1 or -3. **9.** $-\frac{1}{3}$.
10. $\frac{4}{3}$. **11.** 7 or $\frac{1}{7}$. **12.** 4 or $-\frac{1}{3}$.
13. -10 or -11. **14.** $\frac{1}{10}$ or $\frac{1}{11}$.
15. $\frac{4}{3}$ or $-\frac{5}{7}$. **16.** $\frac{4}{7}$ or $-\frac{5}{3}$. **17.** $-\frac{2}{3}$ or $-\frac{3}{2}$.
18. 1 or $-\frac{1}{6}$. **19.** 3 or $-1\frac{17}{29}$. **20.** 2 or $-2\frac{11}{29}$.
21. $\frac{4}{3}$ or $5\frac{2}{3}$. **22.** $\frac{1}{4}$ or $\frac{3}{4}$. **23.** 11 or -21.
24. 0 or 4. **25.** 2 or $\frac{1}{2}$. **26.** $3\pm 2\sqrt{2}$.
27. 3 or -2. **28.** $-\frac{1}{3}$ or $-2\frac{2}{3}$. **29.** 3 or $-4\frac{2}{3}$.
30. 2 or $1\frac{3}{5}$. **31.** $2\pm\sqrt{3}$. **32.** 5 or $1\frac{1}{17}$.
33. 2 or $\frac{1}{2}$. **34.** -1 or $-\frac{16}{3}$. **35.** $2\pm 2\sqrt{3}$.
36. $\frac{-1\pm\sqrt{3}}{2}$. **37.** $\frac{1}{2}$ or -3. **38.** 7 or $1\frac{1}{2}$.
39. 5 or $-\frac{1}{6}$. **40.** $-2\pm\frac{1}{\sqrt{3}}$. **41.** a or $2a$.
42. a or $3a$. **43.** $\frac{-a\pm b}{2}$. **44.** $\frac{a\pm b}{3}$.
45. b or $2a-b$. **46.** b or $-2a-b$.
47. $2a-b$ or $-b$. **48.** c or $c-2b$.
49. 1 or $\frac{a-b}{b-c}$. **50.** 1 or $-\frac{a+b+c}{a+b}$.

51. $\frac{a}{b}$ or $\frac{b}{a}$. **52.** $-\frac{b}{c}$ or $-\frac{c}{b}$.

53. $\frac{a+b}{a-b}$ or $\frac{b-a}{a+b}$. **54.** $\frac{b+a}{b-a}$ or $\frac{b-a}{b+a}$.

55. a or $\frac{1}{a}$. **56.** $\pm\frac{a}{\sqrt{2}}$.

57. $a+b$ or $\frac{2ab}{a+b}$. **58.** $a+b$ or $\frac{ab(a+b)}{a^2+b^2}$.

59. $a+b$ or $\frac{a^2+b^2}{a+b}$. **60.** b or $\frac{a^2}{b}$.

61. c or $-\frac{a^2+ac+b^2+bc}{a+b+2c}$.

62. c or $-\frac{ab(a+b+2c)}{2ab+ac+bc}$. **63.** 0 or $-\frac{1}{2}(a+b)$.

XXXVII. **1.** (i) $(x-2)(x+2)=0$, (ii) $x(x-3)(x+4)=0$,
(iii) $(x+\frac{1}{2})(x+\frac{1}{3})=0$, (iv) $(x-a)(x+b)(x-c)=0$,
(v) $(x-2+\sqrt{2})(x-2-\sqrt{2})=0$,
(vi) $(x-a-\sqrt{b})(x-a+\sqrt{b})=0$.

2. (i) 1, (ii) -2, (iii) $-\frac{7}{3}$. **3.** (i) 0, (ii) -3, (iii) $-\frac{7}{6}$.

4. 12. **5.** p^2-2q. **8.** $\frac{4}{3}$. **9.** 3 or -5.

15. $\frac{b^2-4ac}{a^2}=\frac{q^2-4pr}{p^2}$. **16.** 15. **17.** 2 or 11.

18. 0 or 4. **19.** 4 or $\frac{8}{3}$. **20.** 9 or $\frac{1}{4}$.

21. 1 or 6. **22.** 1 or $-\frac{1}{2}$. **23.** 0 or $\frac{16}{13}$.

24. 5 or $\frac{3}{4}$. **25.** 9 or $-\frac{18}{5}$.

XXXVIII. **1.** $\pm 1, \pm 2$. **2.** $\pm 1, \pm 3$. **3.** $\pm a, \pm\frac{1}{a}$.

4. $\pm a, \pm\frac{1}{a}, \pm\sqrt{-a^2}, \pm\sqrt{\left(-\frac{1}{a^2}\right)}$.

5. $3, -1, \pm 2$. **6.** $4, 1, -2, -5$.

7. $2, -3, \frac{-1\pm\sqrt{-27}}{2}$. **8.** $\frac{1\pm\sqrt{5}}{2}$.

9. $1, \frac{1\pm\sqrt{-15}}{4}$. **10.** $\frac{-1\pm\sqrt{(-1)}}{2}, \frac{-1\pm\sqrt{5}}{2}$.

11. $1, -3 \pm 2\sqrt{2}$. **12.** $\frac{7}{2}, -\frac{3}{2}, 1 \pm \sqrt{5}$.

13. $\dfrac{-3 \pm \sqrt{45}}{2}, \dfrac{-3 \pm \sqrt{17}}{2}$. **14.** $3, 2, \pm\frac{1}{2}$.

15. $5, -8, \dfrac{-3 \pm \sqrt{221}}{2}$. **16.** $1, \dfrac{-1 \pm \sqrt{5}}{2}$.

17. $1, \dfrac{-1 \pm \sqrt{17}}{2}$. **18.** $3, 5, -8$.

19. $3, \pm\sqrt{7}$. **20.** $-1, -1, 4$.

21. $2, \frac{1}{2} \pm \sqrt{(\frac{33}{20})}$. **22.** -3.

XXXIX. **1.** $x=5, y=1$. **2.** $x=7, y=-3$; or $x=3, y=-7$.

3. $x=3, y=\frac{1}{3}$; or $x=\frac{1}{3}, y=3$.

4. $x=5, y=-2\frac{1}{3}$; or $x=-3\frac{1}{2}, y=3\frac{1}{3}$.

5. $x=3, y=\frac{3}{4}$; or $x=-\frac{5}{4}, y=5$.

6. $x=3, y=1$; or $x=\frac{5}{4}, y=-\frac{5}{2}$.

7. $x=\frac{4}{3}, y=\frac{2}{3}$; or $x=\frac{1}{2}, y=\frac{3}{2}$.

8. $x=1, y=3$; or $x=\frac{21}{5}, y=\frac{7}{5}$.

9. $x=3, y=2$; or $x=2, y=3$.

10. $x=3, y=2$; or $x=\frac{2}{5}, y=-\frac{3}{5}$.

11. $x=5, y=2$; or $x=3, y=4$.

12. $x=1, y=3$; or $x=\frac{5}{4}, y=2$.

XL. **1.** $0, 5; 0, -5; \pm 6, \pm 3$.

2. $\pm 3, \pm 2; 0, \sqrt{34}, 0, -\sqrt{34}$.

3. $\pm 3, \pm 4$. **4.** $\pm\dfrac{(a-b)^2}{a+b}, \pm\dfrac{4ab}{a+b}$.

5. $\pm 4, \pm 2; \pm 6\sqrt{-2}, \mp 8\sqrt{-2}$.

6. $\pm 5, \pm 1; \pm 4, \pm\frac{1}{2}$. **7.** $\pm 14, \mp 8; \pm 1, \pm 5$.

8. $\pm 1, \pm 2; \pm\frac{7}{2}\sqrt{-2}, \mp\frac{3}{2}\sqrt{-2}$. **9.** $\pm 5, \pm 2$.

10. $\pm 2, \pm 1$. **11.** $\pm 3, \pm 5; \pm 36, \mp 11\frac{1}{2}$.

12. $\pm 5, \pm 3; \pm 4\sqrt{2}, \pm\sqrt{2}$. **13.** $\pm 1, \pm 2; \pm\sqrt{3}, 0$.

14. $\pm 5, \pm 3$. **15.** $3, -4; -4, 3$. **16.** $3, -6; -6, 3$.

17. $-4, 6$. **18.** $6, 10; 4, 15$.

19. $3 \pm \sqrt{6},\ 3 \mp \sqrt{6}$. **20.** $3,\ -\frac{1}{3};\ -1,\ 1$.

21. $7, 4;\ -4,\ -7$. **22.** $5, 3;\ -3,\ -5$.

23. $4, 3;\ 3, 4$. **24.** $5,\ -4;\ -4,\ 5$.

25. $\pm 3,\ \pm 1;\ \pm 1,\ \pm 3$. **26.** $\pm 3,\ \pm 3$.

27. $4,\ -3;\ -3,\ 4;\ \dfrac{1 \pm \sqrt{5}}{2},\ \dfrac{1 \mp \sqrt{5}}{2}$.

28. $6,\ -1;\ -1,\ 6;\ \dfrac{5 \pm \sqrt{17}}{2},\ \dfrac{5 \mp \sqrt{17}}{2}$.

29. $\frac{3}{2},\ \frac{2}{3}$. **30.** $\frac{6}{7},\ \frac{7}{12};\ \frac{12}{7},\ \frac{7}{6}$.

31. $0, 0;\ 1, 2;\ \frac{15}{22},\ \frac{9}{22}$. **32.** $2, 1;\ \frac{1}{2},\ \frac{1}{8};\ -1,\ 0$.

33. $x = \sqrt[3]{125},\ y = \sqrt[3]{-27}$. **34.** $4, 5;\ 5, 4$.

35. $\pm \frac{1}{2},\ \pm \frac{1}{3};\ \pm \dfrac{\sqrt{-14}}{3},\ \mp \dfrac{\sqrt{-14}}{2}$.

36. $a+b,\ a+b;\ \dfrac{ab-b^2}{a},\ \dfrac{ab-a^2}{b}$. **37.** $a,\ b$.

XLI. **1.** $\pm 27,\ \pm 9$. **2.** 7, 11. **3.** ± 25 and ± 5.

4. 12 and 13. **5.** 38, 42. **6.** 3 or 63.

7. 39 and 29. **8.** ± 2. **9.** 50. **10.** 18.

11. 18 gals. of water. **12.** 20. **13.** 20.

14. 75. **15.** $3\frac{9}{17}$. **16.** 4 miles per hour.

17. 45 and 60. **18.** $1\frac{5}{6}$ hrs. **19.** 4 days.

20. 9*s*. per doz. **21.** 9*d*. per doz. **22.** 48.

23. 7*s*. and 8*s*. 9*d*. **24.** 72. **25.** £10, £5, £1000.

26. $\frac{1}{2}$ and $\frac{1}{3}$, or $\frac{5}{2}$ and $-\frac{5}{3}$. **27.** $3\frac{1}{3}$ and $1\frac{2}{3}$ miles an hr.

28. 578. **29.** 18 in., 8 in. **30.** 1500 sq. yds.

31. 9 and 4. **32.** 210 miles or 144 miles.

MISCELLANEOUS EXAMPLES. IV.

A. **1.** y. **2.** $a^3 - 125b^3 + 8c^3 + 30abc$.

3. $x^2 + x(a-2b) + (a^2 + 3b^2)$.

4. (i) $3(x+y+z)(x-y-5z)$, (ii) $(x^2-1)(y^2-1)$,
(iii) $(x^2z-1)(y^2z-1)$.

5. (i) $\dfrac{x+6}{2-x-x^2}$, (ii) $\dfrac{2}{(x-1)(x-2)(x-3)}$.

6. (i) $x=\dfrac{3ab-2ac-bc}{a+2b-3c}$, (ii) $x=7$ or $-\frac{3}{4}$,
(iii) $x=2$ or $\frac{1}{2}$, $y=\frac{1}{2}$ or 2. **7.** 17 and 18.

B. **1.** (i) 6. **3.** $x^4+2ax^3+3a^2x^2+2a^3x+a^4$.
4. $(64x^6-729)(3x+2)$. **5.** (i) $\dfrac{2(x+3)}{x^4-1}$, (ii) 1.
6. (i) $x=-\frac{1}{5}$, (ii) $x=5$ or $-6\frac{2}{5}$,
(iii) $x=\pm 2$ or $\pm\dfrac{5}{\sqrt{2}}$, $y=\pm 3$ or $\mp\dfrac{1}{\sqrt{2}}$.
8. 840 tickets.

C. **1.** (i) $24a-21b$, (ii) $a^6-6a^4b^2-7a^2b^4-4b^6$.
3. $4xy-6x^2-2y^2$.
4. (i) $(3x+2)(3x+1)$, (ii) $(a+b+c+d)(a-b+c-d)$
$(a+b-c-d)(-a+b+c-d)$.
5. y. **6.** (i) $x=-212$, (ii) $x=y=a$. **7.** 3 and 5.
8. 6*s*. 2*d*., 5*s*. 5*d*.

D. **2.** $3+2x+8x^2+4x^3+x^4$.
5. $(x+1)(x+2)(x+3)(x-7)$, $x=7$.
6. (i) $x=\frac{2}{3}$, $y=3$, (ii) $x=a$ or b,
(iii) $x=2$, $y=1$; $x=6\frac{1}{4}$, $y=-3\frac{1}{4}$. **8.** 10 gallons.

E. **1.** $2\frac{5}{6}$. **3.** $-3a+3b-18c+6d$.
4. $(6x^2-5ax-6a^2)(2x^2+2ax+3a^2)$. **5.** $\dfrac{1}{x+y}$.
6. (i) $x=\dfrac{a+b}{2}$, (ii) $x=0$ or 4,
(iii) $x=\frac{3}{5}$, $y=\frac{2}{3}$; $x=1\frac{1}{2}$, $y=1\frac{2}{3}$. **7.** $x=c$ or $\dfrac{a^2}{c}$.
8. 15 miles an hr.

F. **1.** $a^3+a^2b+ab^2+b^3$. **3.** $1+4x-x^2$. **4.** $x-1$, 1.
5. (i) 0, (ii) x. **6.** (i) $x=1$, (ii) $x=a$ and $y=b$,
(iii) $x=0$ or $\frac{5}{3}$.
7. $7\sqrt{13}$. **8.** 169 or 144.

XLII. **1.** a^{15}. **2.** a^{15}. **3.** $-a^{6}$. **4.** a^{6}.

5. $16a^{20}$. **6.** $-a^{20}$. **7.** $a^{2}b^{8}$. **8.** $a^{15}b^{20}$.

9. $-a^{7}b^{28}$. **10.** $-27a^{21}b^{15}c^{3}$. **11.** $a^{4}b^{8}c^{20}$.

12. $-125a^{6}b^{9}c^{12}$. **13.** $\frac{a^{6}}{b^{6}}$. **14.** $-\frac{a^{15}}{b^{5}c^{10}}$.

15. $\frac{a^{6}}{b^{12}c^{18}}$. **16.** $4a^{8}+12a^{4}b^{3}+9b^{6}$.

17. $a^{10}-4a^{5}b^{4}+4b^{8}$. **18.** $a^{2}x^{8}-2abx^{4}y^{3}+b^{2}y^{6}$.

19. $a^{4}-4a^{3}b+4a^{2}b^{2}$. **20.** $a^{6}+3a^{4}b^{2}+3a^{2}b^{4}+b^{6}$.

21. $a^{9}+3a^{6}b^{3}+3a^{3}b^{6}+b^{9}$.

22. $8a^{6}-36a^{4}b^{2}+54a^{2}b^{4}-27b^{6}$.

23. $27a^{6}-54a^{4}b^{2}+36a^{2}b^{4}-8b^{6}$.

24. $a^{4}+b^{4}+c^{4}+2b^{2}c^{2}+2c^{2}a^{2}+2a^{2}b^{2}$.

25. $a^{6}+4b^{6}+9c^{6}-4a^{3}b^{3}+6c^{3}a^{3}-12b^{3}c^{3}$.

26. $a^{2}+4b^{2}+9c^{2}+16d^{2}+4ab+6ca+8ad+12bc$
$+16bd+24cd$.

27. $1+2x+3x^{2}+4x^{3}+3x^{4}+2x^{5}+x^{6}$.

XLIII. **1.** $3x-5y^{2}$. **2.** $2x^{5}-3y^{3}$. **3.** $x^{4}-3y^{6}$.

4. $\frac{1}{2}x^{4}-\frac{1}{3}y^{5}$. **5.** $a+2b+3c$. **6.** $2a-b-3c$.

7. $2a^{2}+b^{2}-c^{2}$. **8.** $5a^{2}-3b^{2}+2c^{2}$.

9. $x^{2}+x+1$. **10.** $2x^{2}-2x-1$.

11. $3x^{2}-6x-6$. **12.** $1-\frac{1}{2}xy-2x^{2}y^{2}$.

13. $2x^{2}+x-\frac{1}{4}$. **14.** $x^{2}-x+\frac{1}{4}$.

15. $x^{2}+xy+y^{2}$. **16.** $4-12x+9x^{2}$.

17. $1+2x+3x^{2}$. **18.** $2x^{2}-x+\frac{1}{2}$.

19. $1-2x-2x^{2}$. **20.** $x^{3}-2x^{2}+x-2$.

21. $3x^{3}-2x^{2}+3x+2$. **22.** $x^{3}-11x+17$.

23. $a-\frac{1}{2}x+4$. **24.** $x^{2}+x(y+z)+yz$.

XLIV. **1.** 4. **2.** $\frac{1}{8}$. **3.** $\frac{1}{4}$. **4.** $\frac{1}{8}$.

5. 1000. **6.** 25. **7.** $a^{2}b^{2}$. **8.** $a^{-1}b^{\frac{1}{4}}$.

9. $a^{2}b^{8}$. **10.** $a^{-6}b^{-1}$. **11.** $a^{\frac{1}{3}}+x^{\frac{1}{2}}+y^{\frac{3}{4}}$.

12. $a^{\frac{3}{2}}y^{\frac{1}{2}}+a^{\frac{1}{3}}y$. **13.** $a^{\frac{2}{3}}x^{\frac{5}{3}}+a^{\frac{3}{2}}y^{\frac{1}{2}}$. **14.** $a^{-\frac{7}{15}}b^{\frac{2}{5}}$.

15. $\sqrt{a}-\dfrac{1}{a^2}$. **16.** $\dfrac{1}{a^4\sqrt[4]{b}}$. **17.** $\dfrac{\sqrt[3]{a^2}}{b}-\dfrac{b}{\sqrt[3]{a^2}}$.

18. $\dfrac{1}{a^3\sqrt{b}}+3\sqrt[3]{\dfrac{a^2}{b^2}}$. **19.** $x-y$.

20. $1-x^{\frac{3}{4}}$. **21.** $a+b$. **22.** x^2-1.

23. $x^n+2x^{\frac{n}{2}}+3+2x^{-\frac{n}{2}}+x^{-n}$.

24. $\frac{1}{16}a^2-\frac{1}{12}a^{\frac{3}{2}}b^{\frac{1}{2}}+\frac{1}{18}ab-\frac{1}{27}a^{\frac{1}{2}}b^{\frac{3}{2}}+\frac{1}{81}b^2$.

25. $x^{\frac{3}{2}}+y^{\frac{3}{2}}$. **26.** $x^n+x^{\frac{n}{2}}y^{\frac{n}{2}}+y^n$.

27. $x^{\frac{4}{3}}-x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$.

28. $x^{\frac{4}{5}}-3x^{\frac{3}{5}}y^{\frac{1}{3}}+9x^{\frac{2}{5}}y^{\frac{2}{3}}-27x^{\frac{1}{5}}y+81y^{\frac{4}{3}}$.

29. $a^2c^{\frac{5}{2}}$. **30.** a^3-64b^2.

32. (i) $x^{\frac{1}{3}}+1$, (ii) $2x^{\frac{1}{3}}-y^{\frac{1}{4}}$, (iii) $a^{\frac{1}{2}}x^{\frac{1}{3}}-a^{\frac{1}{3}}x^{\frac{1}{2}}$.

33. $2xa^{-1}-3+4x^{-1}a$. **34.** $x^{\frac{1}{3}}-2x^{\frac{1}{2}}+x^{\frac{5}{6}}$.

XLV. **1.** $7\sqrt{3}$. **2.** $8\sqrt[3]{5}$. **3.** $3\sqrt{5}$. **4.** $4\sqrt{2}$.

5. 0. **6.** 30. **7.** 6. **8.** 6.

9. $10\sqrt[6]{40}$. **10.** $4\sqrt[6]{2}$. **11.** $\frac{1}{5}\sqrt{15}$.

12. $\frac{1}{3}\sqrt{6}$. **13.** $3-2\sqrt{2}$. **14.** $\frac{1}{2}(7+3\sqrt{5})$.

15. $5-\sqrt{15}$. **16.** $\frac{1}{71}(103+80\sqrt{3})$.

17. $\frac{1}{4}(2+\sqrt{2}-\sqrt{6})$. **18.** $\frac{3}{22}(4+3\sqrt{3}+\sqrt{5}-2\sqrt{15})$.

19. $2\sqrt[3]{4}+2$. **20.** $\sqrt[3]{81}+1$.

21. 5·828. **22.** 6·635. **23.** $1+\sqrt{5}$.

24. $3+\sqrt{7}$. **25.** $3-\sqrt{3}$. **26.** $5-\sqrt{3}$.

27. $2\sqrt{13}-7$. **28.** $6\sqrt{2}-3\sqrt{5}$.

29. $14+2\sqrt{21}$. **30.** $2+\frac{1}{2}\sqrt{2}$.

31. $2\sqrt[4]{5}$. **32.** $\sqrt{3}-1$.

33. $\sqrt{(a+x)}+\sqrt{(a-x)}$. **34.** $\sqrt{(x+2)}+\sqrt{(2x-3)}$.
35. $\frac{1}{2}(3-\sqrt{7})$. **36.** $\sqrt{8}-\sqrt{7}$.
37. $1+\sqrt{2}+\sqrt{3}$.

XLVI. **1.** 7 : 8, 31 : 36, 41 : 48, 5 : 6. **2.** 2.
3. -13. **4.** $\frac{4}{7}$. **5.** 55 and 66.
6. 21 and 56. **7.** $\frac{7}{2}$. **8.** $\frac{1}{2}$.
9. 2 or 3. **10.** 4 : 7. **11.** 9 and 12.
12. 16. **13.** 5 : 8, 3 : 5.
14. $4x^2 : 9y^2$. **15.** $\frac{11}{16}$.

XLVII. **6.** a^2b^2. **7.** a^2-b^2. **8.** a^3, $(a+b)^3$.

XLVIII. **1.** $8\frac{1}{3}$. **2.** 5. **6.** 6. **7.** $6\frac{3}{4}$.
8. 452·38896 sq. feet. **9.** 113·076. **10.** 160.
11. 64 feet. **13.** $\frac{45}{56}$ cubic feet.
14. $a^2-b^2=4c^2$. **15.** $2053\frac{1}{3}$. **16.** 12 inches.
17. 15 miles.

MISCELLANEOUS EXAMPLES. V.

A. **1.** $\frac{13}{18}$. **2.** $a^2-3ab+b^2$.
3. $1-x^3-x^4$. **4.** $\dfrac{(a-1)(b-1)}{b}$.
5. 3 shillings. **7.** 7.

B. **1.** $3\frac{3}{7}$.
2. $(a-1)x^3-3ax^2+(a^2-a^3)x-(a^3+2a^2+2a+1)$.
3. (i) $x(x-7y)(x-6y)$, (ii) $(5c-a)(3a+4b+c)$,
(iii) $(x-2y+3)(x-2y-3)$.
4. $\dfrac{4x}{x+2y}$. **5.** (i) 4 or $-\frac{5}{3}$;
(ii) $x=\frac{17}{3}$, $y=\frac{10}{3}$; $x=\frac{277}{32}$, $y=\frac{235}{32}$.
7. $3x^3-2xy^2+5y^3$. **8.** 6 and 3.

C. **1.** $20b-10a-8c$. **2.** $b^2-a^2+\frac{b^4}{a^2}-\frac{a^4}{b^2}$.

3. $3\frac{a^2+b^2}{a+b}$, $3a^2+ab-4b^2$. **5.** $a^{\frac{2}{3}}+2a^{\frac{1}{3}}y^{\frac{1}{5}}+3y^{\frac{2}{5}}$.

7. 378 miles. [The word *faster*, in the second line of the question, should be *slower*.]

D. **2.** $x^2+(a-b)x-ab$.

3. $6x(x+1)(x-3)(x-4)$. **4.** $\frac{1}{2}(y+z-x)^2$.

5. (i) $x=-7$; (ii) $x=\pm\frac{2}{7}\sqrt{91}$, $y=\pm\frac{2}{91}\sqrt{91}$.

6. $16\frac{1}{2}$ feet and $13\frac{3}{4}$ feet. **7.** $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

E. **1.** $\frac{1}{3}(x-y)$. **2.** $2ax-(3b-4c)y$.

3. (i) $4a^2b^2(b+2a)(b-2a)$, (ii) $(x+3)(x-29)$,
(iii) $(3x+5y)(x-2y)$.

5. 42. **6.** $2\sqrt{ax}-2a$.

F. **1.** $a^2-b^2-c^2-d^2+2bc+2cd-2bd$.

2. $(a+b+c)^2$. **3.** x^2-5x+6; 3 or 2.

5. (i) $x=-\frac{35}{9}$, (ii) $x=2\frac{1}{2}$, $y=-1\frac{1}{6}$; $x=-1\frac{3}{4}$, $y=1\frac{2}{3}$.

6. $30+13\sqrt{6}$, $y^{-\frac{5}{12}}$. **7.** $x^6-3x^4+2x^2-1$.

XLIX. **1.** (i) 61, (ii) 117, (iii) -75, (iv) $6\frac{5}{9}$, (v) $a-28b$.

2. 34. **3.** -2. **4.** 78. **5.** 0.

6. 21st. **7.** 102nd. **8.** (i) 10, (ii) 0, (iii) a.

9. (i) $\frac{6}{7}$, $\frac{5}{7}$, $\frac{4}{7}$, etc., (ii) $3\frac{1}{3}$, $3\frac{2}{3}$, 4, etc.,
(iii) $\frac{4a}{3}$, $\frac{5a}{3}$, $2a$, etc.

10. $4a-5b$, $3a-4b$, $2a-3b$, etc.

11. $\frac{1}{3}a-4\frac{1}{3}b$, $-\frac{1}{3}a-3\frac{2}{3}b$, $-a-3b$, etc. **13.** 5.

L. **1.** 420. **2.** 180. **3.** $94\frac{1}{2}$. **4.** 660.

5. $3\frac{1}{2}$. **6.** -21. **7.** $30\frac{1}{2}$. **8.** $7\frac{1}{2}$.

9. $\frac{5n(9-n)}{12}$. **10.** $\frac{n(17+7n)}{21}$.

11. $\frac{7(3\sqrt{2}+4)}{\sqrt{2}+1}$. **12.** $\frac{n-1}{2}$.

13. $n(a^2+3ab+b^2-abn)$. **14.** $369\frac{1}{16}$.

15. $6\frac{1}{2}$, 8, $9\frac{1}{2}$, 11, $12\frac{1}{2}$, etc.; $852\frac{1}{2}$.

16. 680. **17.** 3. **18.** 48. **19.** 90.

22. 7500. **23.** 247950. **24.** 3874.

25. $\frac{2n^3-n-1}{2}$.

LI. **1.** (i) $\frac{1}{27}$, (ii) $-\frac{243}{16}$, (iii) $\frac{b^5}{a^3}$. **2.** $\frac{16}{3}$.

3. $-\frac{1}{128}$. **4.** (i) ± 6, (ii) ± 42, (iii) $\pm a^2b^2$.

5. -2, 4; ± 6, 3, $\pm\frac{3}{2}$; 1, $\frac{3}{2}$, $\frac{9}{4}$, $\frac{27}{8}$.

LII. **1.** $16\{(\frac{3}{2})^n-1\}$. **2.** $39\frac{117}{256}$.

3. $\frac{a}{r^{n-1}}\cdot\frac{r^n-1}{r-1}$. **4.** $\frac{27}{2}\{1-(\frac{8}{9})^8\}$.

5. $4\frac{89}{144}$. **6.** $6\frac{6}{7}$. **7.** $5\frac{5}{7}$.

8. 5. **9.** $\frac{b^n-a^n}{a^{n-3}(b-a)}$. **10.** $1\frac{4}{5}$.

11. 8190. **12.** 3·4142. **13.** $6x^2-5x-6$.

15. 3, $\frac{3}{4}$, &c., or 1, $\frac{3}{4}$, &c. **16.** 12, -4, &c. **17.** 2.

LIII. **1.** 0. **2.** 1195742. **3.** $1\frac{1}{4}$.

4. $\frac{b^n-a^n}{a^{n-2}(b-a)}$. **5.** $\frac{1}{3}$. **6.** $104\frac{1}{2}$.

7. $76a+57b$. **8.** $\frac{3}{28}\{1-(\frac{4}{3})^8\}$. **9.** $5\frac{5}{24}$.

10. $161\frac{3}{32}$. **11.** (i) $71\frac{1}{4}$;
(ii) -6; (iii) $\frac{16}{7}\{1-(\frac{3}{4})^6\}$, $\frac{16}{7}$; (iv) $6\frac{89}{96}$;
(v) $\frac{3}{2}\{1-(\frac{2}{3})^6\}$, $\frac{3}{2}$.

13. 1, 3, 5, &c. **14.** $5\pm 2\sqrt{6}$.

17. 2, 6, 18. **18.** 1, 3, 5, 7, 9.

19. 9 and 12, or 1 and -4.

LIV. **2.** $\frac{7}{6}$, $\frac{7}{5}$, $\frac{7}{4}$, $\frac{7}{3}$, $\frac{7}{2}$. **3.** $\frac{3}{4}$, $\frac{6}{7}$, 1, $\frac{6}{5}$.

LV. **1.** $\frac{1}{2}\left(1-\frac{1}{2n+1}\right);\ \frac{1}{2}$. **2.** $\frac{1}{4}\left(1-\frac{1}{n+1}\right);\ \frac{1}{4}$.

3. $\frac{1}{4}\left\{\frac{1}{3}-\frac{1}{(2n+1)(2n+3)}\right\};\ \frac{1}{12}$.

4. $\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right);\ \frac{1}{6}$.

5. $\frac{1}{x+1}-\frac{1}{x+n+1};\ \frac{1}{x+1}$.

6. $\frac{1}{x}\left\{\frac{1}{1+x}-\frac{1}{1+(n+1)x}\right\};\ \frac{1}{x(1+x)}$.

7. $\frac{4}{3}n(n+1)(n+2)$.

8. $\frac{1}{4}.n(n+1)(n+2)(n+3)$.

9. $\frac{1}{2}+\frac{1}{6}(2n-1)(2n+1)(2n+3)$.

10. $\frac{1}{3}n(4n^2-1)$. **11.** $\frac{2}{3}.n(n+1)(2n+1)$.

12. $na^2+n(n-1)ab+\frac{1}{6}n(n-1)(2n-1)b^2$.

13. $na^2+n^2ab+\frac{1}{3}n(n-1)(n+1)b^2$.

MISCELLANEOUS EXAMPLES. VI.

A. **1.** 48. **3.** $3x-7,\ x=\frac{7}{3}$.

4. (i) $x=-4,\ y=\frac{1}{7}$, (ii) $x=c$, or $x=a+b+c$.

5. £20 each ox, £2. 10*s*. each sheep.

7. $ab^{-1}+32a^{-\frac{2}{3}}b^{\frac{3}{2}}$. **8.** $-\frac{1}{23}(73+40\sqrt{3}),\ \sqrt{8}+\sqrt{17}$.

9. $\frac{20}{7}\{1-(\frac{3}{4})^8\},\ \frac{20}{7}$.

10. $109\frac{11}{21},\ 119\frac{1}{21},\ \ldots\ 290\frac{10}{21}$, sum 4000.

B. **1.** $x-8y+4z$. **2.** $(nx+my)(mx+ny),\ (x-y)(x^2+y^2)$.

4. (i) $x=\pm\sqrt{ab}$, (ii) $x=7$ or -2. **6.** a^3+b^2-c.

8. (i) -1040, (ii) $4\frac{1}{6}$. **9.** 71071. **10.** 88 yards.

C. **1.** 2. **2.** $2x^2-3xy+2y^2$. **4.** $3(x-3a)^2(x^2-4a^2)$.

5. $\frac{6a^2}{(a^2-4)(a^2-1)}$. **6.** 25 miles an hour, 100 miles.

7. 1. **9.** (i) 0, (ii) $8\frac{1}{3}$. **10.** 9 and 25.

D. **1.** $x^4 - a^4$. **3.** $\frac{(x-4)(x-7)}{x^2}$.

4. (i) $x = \frac{1}{2}$, $y = -\frac{1}{3}$, (ii) $x = \pm 5$, $y = \pm 3$; $x = \pm \frac{13}{3}$, $y = \pm \frac{5}{3}$.

5. 48. **6.** $x^2 - 40x + 8 = 0$. **7.** $x^3 - x^{-1} + 4x^{-3}$.

9. (i) $-7\frac{353}{1536}$, (ii) $5\frac{2}{5}$.

E. **1.** $-x^2 - xy - y^2$. **2.** $3a^2 + 4ab + b^2$.

3. $a - b$, $(a-b)(4a-b)(3a^2+b^2)$. **4.** $(x+1)^2$.

5. $-mn$. **6.** $x = -\frac{5}{6}$. **8.** $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}}y^{\frac{1}{2}} - 4y$.

9. (i) $\frac{1}{2}n(n-7)$, (ii) $\frac{2}{3}$. **10.** $22\frac{1}{2}$ and 45.

F. **1.** $8xz$. **2.** $a^2 + b^2 + c^2 - 2ab + ac - bc$. **4.** $-\frac{2}{a+2x}$.

5. (i) $x = 3$, $y = 7$, $z = 4$. (ii) $x = 4\frac{1}{2}$. (iii) $x = \pm 9$, $y = \pm 3$.

6. Expression is $(x^3 - 2x^2 + 5x - 6)^2$. **7.** $\frac{1}{2}(\sqrt{5} + \sqrt{7})$.

LVI. **1.** 2730, 5040, 40320. **3.** 420, 34650, 172972800.

4. 1260. **5.** 120, 210. **6.** 720. **7.** 8.

8. 1820, 3003, 6. **9.** 105. **10.** 5. **15.** 210.

16. 364. **17.** 60. **18.** 90. **19.** 420. **20.** 9.

LVII. **1.** $x^6 + 6x^5a + 15x^4a^2 + 20x^3a^3 + 15x^2a^4 + 6xa^5 + a^6$.

2. $1 - 5x^2 + 10x^4 - 10x^6 + 5x^8 - x^{10}$.

3. $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$.

4. $16a^4 + 96a^5 + 216a^6 + 216a^7 + 81a^8$.

5. $64x^{12} - 192x^{10} + 240x^8 - 160x^6 + 60x^4 - 12x^2 + 1$.

6. $y^7 - 7y^6x + 21y^5x^2 - 35y^4x^3 + 35y^3x^4 - 21y^2x^5 + 7yx^6 - x^7$.

7. $405a^8b^2$. **8.** $126720x^{16}$. **9.** $-14a^3$.

10. $210x^6$. **11.** $1140x^{17}$. **12.** $231x^{20}$.

13. $2a^4 + 12a^2b + 2b^2$. **14.** $2a^6 + 30a^4b + 30a^2b^2 + 2b^3$.

15. $2a^8 + 56a^6b + 140a^4b^2 + 56a^2b^3 + 2b^4$. **16.** $70x^4$.

17. $252x^5$. **18.** $90720x^4y^4$. **19.** $70a^4x^4$.

20. $126a^5x^4$ and $126a^4x^5$.

21. a^{12}, $12a^{11}x$, $66a^{10}x^2$; $66a^2x^{10}$, $12ax^{11}$, x^{12}.

24. 96059601, 997002999.

LIX. **1.** $(x+1)(x-1)^2$. **2.** $(x^2+1)(x+1)^2(x-1)^2$.

3. $(1-x)^2(1+x+x^2)$.

4. $(1-x)^2(1+x)^2(1+x+x^2)(1-x+x^2)$.

5. $(x-y-a-b)(x+y-a+b)$. **6.** $(x+y-1)(x-y-2)$.

7. $(x+2y-4)(x-4y+2)$. **8.** $(a+b-c+f)(a-b-c-f)$.

9. (i) $x^4+x^3y+x^2y^2+xy^3+y^4$.

(ii) $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.

(iii) $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.

12. $(2a+3b)^2-(2a+3b)(3a+2b)+(3a+2b)^2$.

13. $3(2a+4b-4c)^2-3(2a+4b-4c)(a-b+7c)+3(a-b+7c)^2$.

LX. **9.** $-(b-c)(c-a)(a-b)(a+b+c)$. **10.** $4abc$.

11. $2abc$. **12.** $2abc(a+b+c)$.

13. 3; $5(x^2+y^2+z^2+yz+zx+xy)$. **15.** $a+b+c$.

16. 1. **17.** $\dfrac{1}{abc}$. **18.** $\dfrac{bc+ca+ab}{a^2b^2c^2}$.

19. 0. **20.** $\dfrac{a+b+c}{abc}$.

24. $(b+c)(c+a)(a+b)$.

45. (i) $x-8y$. (ii) $1+x+x^2$. (iii) x^2-3x+2. (iv) $2x^2-3x+4$.

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